O-LEVEL A-MATHS 2012 – PAPER 2

Question 1

(i) $y = 2\sin x - 3\cos x$

$\frac{dy}{dx} = 2\cos x + 3\sin x$

$\frac{d^2y}{dx^2} = -2\sin x + 3\cos x$

(ii) At stationary point,

$\frac{dy}{dx} = 0$

$2\cos x + 3\sin x = 0$

$3\sin x = -2\cos x \Rightarrow \tan x = -\frac{2}{3}$

Basic $\angle = \tan^{-1}\frac{2}{3}$

$x = \pi - \tan^{-1}\frac{2}{3} = 2.55$

(iii) When $x = \pi - \tan^{-1}\frac{2}{3}$,

$\frac{d^2y}{dx^2} = -2\sin\left(\pi - \tan^{-1}\frac{2}{3}\right) + 3\cos\left(\pi - \tan^{-1}\frac{2}{3}\right)$

$= -3.61 < 0$

∴ This is a maximum point.

Question 2

(i) $\cos 3x + \cos x$

$= 2\cos\frac{1}{2}(3x + x)\cos\frac{1}{2}(3x - x)$

$= 2\cos 2x \cos x$

$\therefore A = 2,\ B = 2$

(ii) $\cos 3x + 2\cos x = 0$

$(2\cos 2x\cos x - \cos x) + 2\cos x = 0$

$2\cos 2x\cos x + \cos x = 0$

$\cos x\,(2\cos 2x + 1) = 0$

$\cos x = 0$ or $\cos 2x = -\frac{1}{2}$

$x = 90°$

Basic ∠

$= \cos^{-1}\frac{1}{2} = 60°$

$0° \le 2x \le 360°$

$2x = 180° - 60°, 180° + 60°$

$x = 60°, 120°$

Question 3

(i) $\frac{20}{3x^2+8x-3}=\frac{20}{(3x-1)(x+3)}$

Let $\frac{20}{3x^2+8x-3}=\frac{A}{3x-1}+\frac{B}{x+3}$

$\therefore 20 = A(x+3)+B(3x-1)$

Let $x=-3$

$20 = B(-9-1) \Rightarrow B=-2$

Let $x=\frac{1}{3}$,

$20 = A\left(\frac{1}{3}+3\right) \Rightarrow A=6$

$\therefore \frac{20}{3x^2+8x-3}=\frac{6}{3x-1}-\frac{2}{x+3}$

(ii) $\int \frac{20}{3x^2+8x-3}dx$

$= \int \frac{6}{3x-1}-\frac{2}{x+3}dx$

$= 6\left(\frac{\ln|3x-1|}{3}\right)-2\ln|x+3|$

$= 2\ln|3x-1|-2\ln|x+3|+c$

$\int_2^7 \frac{20}{3x^2+8x-3}dx$

$= [2\ln|3x-1|-2\ln|x+3|]_2^7$

$= (2\ln 20-2\ln 10)-(2\ln 5-2\ln 5)$

$= 2\ln 20-2\ln 10$

$= 2\ln\left(\frac{20}{10}\right)$

$= 2\ln 2$

Question 4

(i) $\angle RPQ=\angle PSQ$ (alternate segment theorem)

$\angle PRQ=\angle SPQ$ (alternate segment theorem)

$\therefore \Delta PQR$ is similar to ΔSQP (AA similarity) (proven)

(ii) By similar triangles,

$\frac{QS}{QP}=\frac{QP}{QR} \Rightarrow QS\times QR=(QP)^2$

(proven)

(iii) $\angle SAP+\angle PQS=180°$ ($\angle$s in opposite segments)

$\angle PBR+\angle RQP=180°$ ($\angle$s in opposite segments)

$\angle PQS=\angle RQP$ (similar Δs)

$\therefore \angle SAP=\angle PBR$ (proven)

Question 5

(i) At $P(0, 4)$,

$$Ae^0 + Be^0 = 4 \Rightarrow A + B = 4 \quad (1)$$

$$\frac{dy}{dx} = 2Ae^{2x} - Be^{-x}$$

At $P(0, 4)$

$$\frac{dy}{dx} = -1$$

$$\Rightarrow 2Ae^0 - Be^0 = -1$$

$$2A - B = -1 \quad (2)$$

$(1) + (2)$

$$3A = 3 \Rightarrow A = 1$$

Sub. $A = 1$ into (1)

$$1 + B = 4 \Rightarrow B = 3$$

(ii) $\int y dx$

$$= \int e^{2x} + 3e^{-x}\, dx$$

$$= \frac{1}{2}e^{2x} - 3e^{-x} + c$$

$$\int_0^1 y\, dx = \left[\frac{1}{2}e^{2x} - 3e^{-x}\right]_0^1$$

$$= \left(\frac{1}{2}e^2 - 3e^{-1}\right) - \left(\frac{1}{2} - 3\right) = 5.1$$

Question 6

(a) (i) $\log_8 x^3 = \log_4 u$

$3\log_8 x = \log_4 u$

$3\left(\frac{\log_2 x}{\log_2 8}\right) = \frac{\log_2 u}{\log_2 4}$

$3\left(\frac{\log_2 x}{3}\right) = \frac{\log_2 u}{2}$

$2\log_2 x = \log_2 u$

$\log_2 u = \log_2 x^2 \Rightarrow u = x^2$

(ii) $\log_4\left(x^2+5x\right) - \log_8 x^3 = \frac{1}{\log_3 4}$

$\frac{\log_2(x^2+5x)}{\log_2 4} - \frac{\log_2 x^3}{\log_2 8} = \frac{1}{\log_2 4 / \log_2 3}$

$\frac{\log_2(x^2+5x)}{2} - \frac{\log_2 x^3}{3} = \frac{\log_2 3}{2}$

$\log_2(x^2+5x) - \frac{2}{3}\log_2 x^3 = \log_2 3$

$\log_2(x^2+5x) - \log_2 x^2 = \log_2 3$

$\log_2\left(\frac{x^2+5x}{x^2}\right) = \log_2 3$

$\frac{x^2+5x}{x^2} = 3$

$x^2+5x = 3x^2$

$2x^2 - 5x = 0$

$x(2x-5) = 0$

$x = 0$ (NA) or $x = \frac{5}{2}$

(b) $e^y(e^y - 2) = 15$

Let $u = e^y$

$u(u-2) = 15$

$u^2 - 2u - 15 = 0$

$(u+3)(u-5) = 0$

$u = -3$ or $u = 5$

$e^y = -3$ (NA) $\quad e^y = 5$

$y = \ln 5$

Question 7

(i) $2x^2 + 3x - 2 = (2x - 1)(x + 2)$

Let $f(x) = 2x^4 + 3x^3 + a(x^2 + x) + b$

$f\left(\frac{1}{2}\right) = 0$

$2\left(\frac{1}{2}\right)^4 + 3\left(\frac{1}{2}\right)^3 + a\left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)\right] + b = 0$

$\frac{1}{2} + \frac{3}{4}a + b = 0$

$\frac{3}{4}a + b = -\frac{1}{2} \Rightarrow 3a + 4b = -2 \quad (1)$

$f(-2) = 0$

$2(-2)^4 + 3(-2)^3 + a[(-2)^2 + (-2)] + b = 0$

$8 + 2a + b = 0 \Rightarrow b = -2a - 8 \quad (2)$

Sub. (2) into (1)

$3a + 4(-2a - 8) = -2$

$-5a = 30 \Rightarrow a = -6$

Sub. $a = -6$ into (2)

$b = -2(-6) - 8 = 4$

(ii) $f(x) = 2x^4 + 3x^3 - 6x^2 - 6x + 4$

Let $2x^4 + 3x^3 - 6x^2 - 6x + 4 = (2x^2 + 3x - 2)(x^2 + Ax - 2)$

Let $x = 1$,

$2 + 3 - 6 - 6 + 4 = (2 + 3 - 2)(1 + A - 2)$

$-3 = (3)(A - 1) \Rightarrow A = 0$

$\therefore 2x^4 + 3x^3 - 6x^2 - 6x + 4 = 0$

$(2x^2 + 3x - 2)(x^2 - 2) = 0$

$(2x - 1)(x + 2)(x - \sqrt{2})(x + \sqrt{2}) = 0$

$x = -2,\ -\sqrt{2},\ \frac{1}{2}$ or $\sqrt{2}$

Question 8

(i) $y = 2x - x^2 \quad (1)$

$y = mx + 1 \quad (2)$

$(1) = (2)$

$2x - x^2 = mx + 1$

$x^2 + (m - 2)x + 1 = 0$

Discriminant < 0

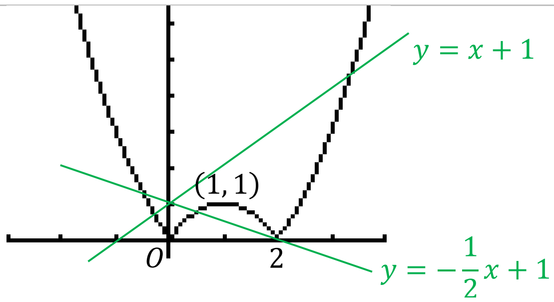
$(m - 2)^2 - 4(1)(1) < 0$

$(m - 2 + 2)(m - 2 - 2) < 0$

$m(m - 4) < 0$

$0 < m < 4$

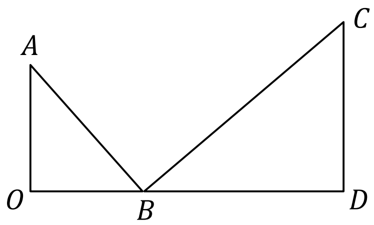
(ii) $y = |2x - x^2|$



(iii) (a) 2 (b) 3

Question 9

(i)



$\angle CBD = \angle OAB = \theta$

$\sin\theta = \frac{OB}{10} \Rightarrow OB = 10\sin\theta$

$\cos\theta = \frac{BD}{24} \Rightarrow BD = 24\cos\theta$

$\therefore L = OB + BD = 10\sin\theta + 24\cos\theta$

(ii) Let

$L = 10\sin\theta + 24\cos\theta$
$= R\cos(\theta - \alpha)$
$= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$\therefore R\cos\alpha = 24 \quad (1)$
$R\sin\alpha = 10 \quad (2)$

$(2)/(1) \quad \tan\alpha = \frac{10}{24} \Rightarrow \alpha = 22.620°$

$(1)^2 + (2)^2 \quad R^2 = 24^2 + 10^2$
$\Rightarrow R = \sqrt{676} = 26$

$\therefore L = 26\cos(\theta - 22.6°)$

(iii) Greatest $L = 26$

This happens when
$\cos(\theta - 22.620°) = 1$
$\theta - 22.620° = 0$
$\theta = 22.6°$

(iv) $L = 20$
$26\cos(\theta - 22.620°) = 20$
$\theta = 22.6199° + \cos^{-1}\frac{20}{26}$
$= 62.3°$

Question 10

(i) Midpoint of AB

$= \left(\frac{3+8}{2}, \frac{6+k}{2}\right) = \left(\frac{11}{2}, \frac{6+k}{2}\right)$

As midpoint of AB lies on $y + 3x = 25$,

$\frac{6+k}{2} + 3\left(\frac{11}{2}\right) = 25$

$\frac{6+k}{2} = \frac{17}{2} \Rightarrow 6 + k = 17$

$\Rightarrow k = 11$ (shown)

(ii) Gradient of BC

$= \frac{6-2}{3-1} = 2$

Equation of AD:

$y - 11 = 2(x - 8)$

$\Rightarrow y = 2x - 5 \quad (1)$

$y + 3x = 25 \quad (2)$

Sub. (1) into (2)

$2x - 5 + 3x = 25$

$\Rightarrow 5x = 30 \Rightarrow x = 6$

Sub. $x = 6$ into (1)

$y = 2(6) - 5 = 7$

$\therefore N(6, 7)$

Equation of CD:

$y - 2 = -\frac{1}{2}(x - 1)$

$\Rightarrow y = -\frac{1}{2}x + \frac{5}{2} \quad (3)$

$(1) = (3)$

$2x - 5 = -\frac{1}{2}x + \frac{5}{2}$

$\frac{5}{2}x = \frac{15}{2} \Rightarrow x = 3$

Sub. $x = 3$ into (1)

$y = 2(3) - 5 = 1$

$\therefore D(3, 1)$

(iii) $M\left(\frac{11}{2}, \frac{17}{2}\right)$

Area of ΔAMN

$= \frac{1}{2}\begin{vmatrix} 8 & {}^{11}/_{2} & 6 & 8 \\ 11 & {}^{17}/_{2} & 7 & 11 \end{vmatrix}$

$= \frac{1}{2}\left[\left(68 + \frac{77}{2} + 66\right) - \left(\frac{121}{2} + 51 + 56\right)\right]$

$= \frac{5}{2}$

Question 11

(i) $y=(5+4x)^{\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{1}{2}(5+4x)^{-\frac{1}{2}}(4)=\frac{2}{\sqrt{5+4x}}$$

When $x=1$

$$y=\sqrt{5+4}=3$$

$$\frac{dy}{dx}=\frac{2}{\sqrt{5+4}}=\frac{2}{3}$$

Equation of normal at B:

$$y-3=-\frac{3}{2}(x-1)$$

$$y=-\frac{3}{2}x+\frac{9}{2}$$

Let $y=0$,

$$-\frac{3}{2}x+\frac{9}{2}=0$$

$$\frac{3}{2}x=\frac{9}{2}\Rightarrow x=3$$

$\therefore C(3,0)$

(ii) For $y=\sqrt{5+4x}$, at A,

$$y=0,$$

$$\sqrt{5+4x}=0$$

$$x=-\frac{5}{4}$$

$$\therefore A\left(-\frac{5}{4},0\right)$$

Area

$$=\int_{-\frac{5}{4}}^{1}(5+4x)^{\frac{1}{2}}\,dx+\frac{1}{2}(2)(3)$$

$$=\left[\frac{(5+4x)^{\frac{3}{2}}}{(4)\left({}^{3}/_{2}\right)}\right]_{-\frac{5}{4}}^{1}+3$$

$$=\frac{1}{6}\left[(5+4x)^{\frac{3}{2}}\right]_{-\frac{5}{4}}^{1}+3$$

$$=\frac{1}{6}\left[(5+4)^{\frac{3}{2}}-(5-5)^{\frac{3}{2}}\right]+3$$

$$=\frac{15}{2}$$