A-LEVEL H2 MATHS 2010 – PAPER 2

Question 1

[Ans: (i) $3 \pm 5i$ (ii) $a = -16$, $b = -20$; $-2 - i$, -2 and 2]

(i) $x^2 - 6x + 34 = 0$

$x = \dfrac{6 \pm \sqrt{6^2 - 4(1)(34)}}{2}$

$x = \dfrac{6 \pm \sqrt{-100}}{2} = 3 \pm 5i$

(ii) Let $f(x) = x^4 + 4x^3 + x^2 + ax + b$

$f(-2 + i) = 0$

$(-2 + i)^4 + 4(-2 + i)^3 + (-2 + i)^2 + a(-2 + i) + b = 0$

$-12 + 16i - 2a + ia + b = 0$

$(-12 - 2a + b) + i(16 + a) = 0$

$\therefore 16 + a = 0$

$\Rightarrow a = -16$

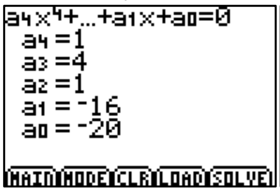
$\therefore -12 - 2a + b = 0$

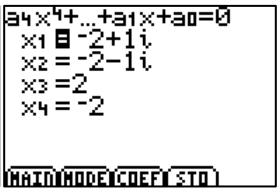
$-12 - 2(-16) + b = 0$

$\Rightarrow b = -20$

$\therefore x^4 + 4x^3 + x^2 - 16x - 20 = 0$

From GC,





The other roots are $-2 - i$, -2 and 2.

Question 2

[Ans: (i) prove (ii)(a) prove (b) explain; $\frac{3}{4}$]

(i) Let

$$p_n: \sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7), n \in \mathbb{Z}^+$$

$$P_1: LHS = \sum_{r=1}^{1} r(r+2) = (1)(1+2) = 3$$

$$RHS = \frac{1}{6}(1)(1+1)(2+7) = 3 = LHS$$

$\therefore P_1$ is true.

Assume P_k is true. i.e.

$$\sum_{r=1}^{k} r(r+2) = \frac{1}{6}k(k+1)(2k+7).$$

[Aim: To prove P_{k+1} is true. i.e.

$$\sum_{r=1}^{k+1} r(r+2) = \frac{1}{6}(k+1)[(k+1)+1][2(k+1)+7] = \frac{1}{6}(k+1)(k+2)(2k+9)]$$

$$\sum_{r=1}^{k+1} r(r+2)$$
$$= \sum_{r=1}^{k} r(r+2) + (k+1)(k+3)$$
$$= \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3)$$
$$= \frac{1}{6}(k+1)[k(2k+7) + 6(k+3)]$$
$$= \frac{1}{6}(k+1)(2k^2+13k+18)$$
$$= \frac{1}{6}(k+1)(k+2)(2k+9)$$

$\therefore P_{k+1}$ is true if P_k is true.

Since P_1 is true, $\therefore$ by mathematical induction, P_n is true for $n \in \mathbb{Z}^+$.

(ii) (a) Let $\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$

$A = \frac{1}{2}$ and $B = -\frac{1}{2}$

$$\therefore \frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$$
$$= \frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+2}\right)$$

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+2}\right)$$
$$= \frac{1}{2}[1 - \frac{1}{3}$$
$$+\frac{1}{2} - \frac{1}{4}$$
$$+\frac{1}{3} - \frac{1}{5}$$
$$+\frac{1}{4} - \frac{1}{6}$$
$$\vdots$$
$$+\frac{1}{n-2} - \frac{1}{n}$$
$$+\frac{1}{n-1} - \frac{1}{n+1}$$
$$+\frac{1}{n} - \frac{1}{n+2}]$$
$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$
$$= \frac{3}{4} + \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$
(shown)

(b) When $n \to 0$, $\frac{1}{2(n+1)} \to 0$ and $\frac{1}{2(n+2)} \to 0$, ∴ series is convergent.

Sum to infinity
$$\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$$
$$= \lim_{n\to\infty}\left[\frac{3}{4} + \frac{1}{2(n+1)} - \frac{1}{2(n+2)}\right]$$
$$= \frac{3}{4}$$

Question 3

[Ans: (i) $\frac{dy}{dx} = \frac{3x+4}{2\sqrt{x+2}}$; show (ii)(a) $\pm\sqrt{2}$ (b) sketch (iii) sketch]

(i) $\frac{dy}{dx} = x\left(\frac{1}{2}\right)(x+2)^{-\frac{1}{2}} + \sqrt{x+2}$

$= \frac{x}{2\sqrt{x+2}} + \sqrt{x+2}$

$= \frac{x + 2x + 4}{2\sqrt{x+2}} = \frac{3x+4}{2\sqrt{x+2}}$

Let $\frac{dy}{dx} = 0$

$\therefore \frac{3x+4}{2\sqrt{x+2}} = 0 \Rightarrow x = -\frac{4}{3}$

$\therefore$ There is only one turning point at $x = -\frac{4}{3}$.

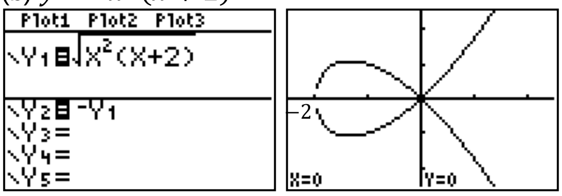
(ii) (a) $y^2 = x^2(x+2) \Rightarrow y = \pm x\sqrt{x+2}$

$\therefore \frac{dy}{dx} = \pm\frac{3x+4}{2\sqrt{x+2}}$

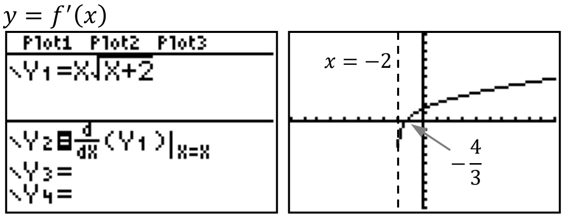
When $x = 0$,

$\frac{dy}{dx} = \pm\frac{4}{2\sqrt{2}} = \pm\sqrt{2}$

(b) $y^2 = x^2(x+2)$



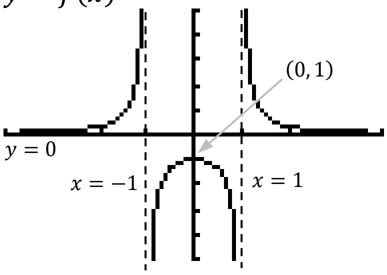
(iii) $y = f'(x)$



Question 4

[Ans: (i) sketch (ii) state; $k = 0$ (iii) show (iv) $2 < x < 3$ or $3 < x < 4$ (v) $(-\infty, -1) \cup (0, \infty)$]

(i) $y = f(x)$



(ii) For f^{-1} to exist, the f must be a one-to-one function. Therefore least value of $k = 0$.

(iii) $fg(x)$

$$= \frac{1}{\left(\frac{1}{x-3}\right)^2 - 1}$$

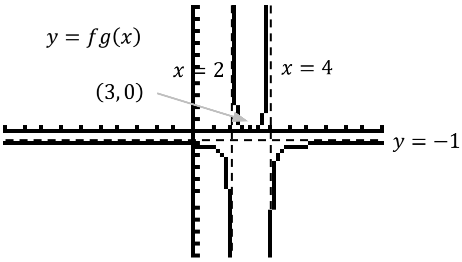
$$= \frac{1}{\frac{1-(x-3)^2}{(x-3)^2}}$$

$$= \frac{(x-3)^2}{(1-x+3)(1+x-3)}$$

$$= \frac{(x-3)^2}{(4-x)(x-2)}$$

(shown)

(iv) $fg(x) > 0$, $x \in D_{fg} = D_g$



From GC,

$2 < x < 3$ or $3 < x < 4$

(v) From graph of $y = fg(x)$,

$\therefore R_{fg} = (-\infty, -1) \cup (0, \infty)$

Question 5

[Ans: (i) reason (ii) explain]

(i) Opinions of international spectators in a stratified sample may not be representative of the population across the various strata due to cultural and regional influences.

(ii) To perform a systematic sampling, we can first determine the size of the sample, $0.01N$, where N is the total number of spectators. Then we will randomly interview a spectator coming out of the catering location, and follow by every interviewing every $\frac{N}{0.01N} = 100^{\text{th}}$ spectators leaving the location.

Question 6

[Ans: $\mu \approx 41.3$, $s^2 = 1.584$; p-value $= 0.0949$, sufficient evidence to reject H_0]

$$\mu \approx \bar{t} = \frac{\sum t}{n}$$
$$= \frac{454.3}{11} = 41.3$$

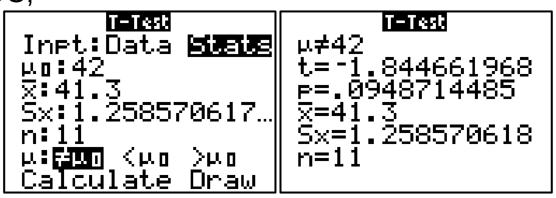
$$\sigma^2 \approx s^2 = \frac{1}{n-1}\left[\sum t^2 - \frac{(\sum t)^2}{n}\right]$$
$$= \frac{1}{10}\left[18778.43 - \frac{(454.3)^2}{11}\right]$$
$$= 1.584$$

$H_0: \mu = 42.0$
$H_1: \mu \neq 42.0$

$n = 11$
$\bar{t} = 41.3$
$s^2 = 1.584$

Test statistics, $T = \dfrac{\bar{T} - 42.0}{\sqrt{1.584/11}} \sim t(10)$

From GC,



p-value $= 0.094871$

Since p-value < 0.10, $\therefore$ there is sufficient evidence to reject H_0, i.e. there has been a change in the mean time required by an employee to complete the task, at 10% level of significance

Question 7

[Ans: (i) 0.32 (ii) 0.92 (iii) 0.457 (iv) 0.15 (v) $0.07 \leq P(A' \cap B \cap C) \leq 0.15$]

(i) $P(A|B') = 0.8$

$\frac{P(A \cap B')}{P(B')} = 0.8$

$P(A \cap B') = 0.8P(B')$

$P(A \cap B') = 0.8[1 - P(B)] = 0.32$

(ii) $P(A \cup B)$

$= P(A \cap B') + P(B)$

$= 0.32 + 0.6 = 0.92$

(iii) $P(B'|A)$

$= \frac{P(B' \cap A)}{P(A)} = \frac{P(A \cap B')}{P(A)}$

$= \frac{0.32}{0.7} = 0.457$

(iv) $P(C) = P(A \cap C) + P(A' \cap C)$

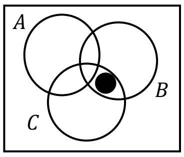
$P(A' \cap C) = P(C) - P(A \cap C)$

$P(A' \cap C) = P(C) - P(A)P(C)$

$P(A' \cap C) = 0.5 - (0.7)(0.5)$

$= 0.15$

(v)



From the Venn diagram,

$P(A' \cap B' \cap C) \leq 1 - P(A \cup B) = 1 - 0.92 = 0.08$

$P(A' \cap C) - P(A' \cap B \cap C) \leq 0.08$

$0.15 - P(A' \cap B \cap C) \leq 0.08$

$\Rightarrow P(A' \cap B \cap C) \geq 0.07$

$P(A' \cap B \cap C) \leq P(A' \cap C) = 0.15$

$\therefore 0.07 \leq P(A' \cap B \cap C) \leq 0.15$

Question 8

[Ans: (i) $\frac{3}{5}$ (ii) $\frac{1}{10}$ (iii) $\frac{7}{20}$]

(i) Required probability
$= \frac{3}{5}$

(ii) _ _ _ E E

Required probability
$= \frac{3!\,2!}{5!} = \frac{1}{10}$

(iii) Case (1): 3 _ _ _ {1 / 5}
No of ways = ${}^{2}C_{1}3! = 12$

Case (2): 4 _ _ _ {1 / 3 / 5 }
No of ways = ${}^{3}C_{1}3! = 18$

Case (3): 5 _ _ _ {1 / 3 }
No of ways = ${}^{2}C_{1}3! = 12$

Required probability
$= \frac{12 + 18 + 12}{5!}$
$= \frac{7}{20}$

Question 9

[Ans: (i) 0.681 (ii) 0.234 (iii) 0.362]

(i) Given $X \sim N(180, 30^2)$ and $Y \sim N(400, 60^2)$

$E(Y - 2X) = E(Y) - 2E(X)$
$= 400 - 2(180) = 40$

$Var(Y - 2X) = Var(Y) + 2^2 Var(X)$
$= 60^2 + 4(30^2) = 7200$

$\therefore\ Y - 2X \sim N(40, 7200)$

Required probability
$= P(Y > 2X) = P(Y - 2X > 0)$
$= 0.681$

(ii) $E(0.12X + 0.05Y)$
$= 0.12E(X) + 0.05E(Y)$
$= 0.12(180) + 0.05(400) = 41.6$

$Var(0.12X + 0.05Y)$
$= 0.12^2 Var(X) + 0.05^2 Var(Y)$
$= 0.12^2(30^2) + 0.05^2(60^2)$
$= 21.96$

$\therefore 0.12X + 0.05Y \sim N(41.6, 21.96)$

Required probability
$= P(0.12X + 0.05Y > 45) = 0.234$

(iii) Let $C = 0.12X$
$E(C) = 0.12E(X)$
$= 0.12(180) = 21.6$
$Var(C) = 0.12^2 Var(X)$
$= 0.12^2(30^2) = 12.96$

$\therefore C \sim N(21.6, 12.96)$

$E(C_1 + C_2) = 2E(C)$
$= 2(21.6) = 43.2$
$Var(C_1 + C_2) = 2Var(C)$
$= 2(12.96) = 25.92$

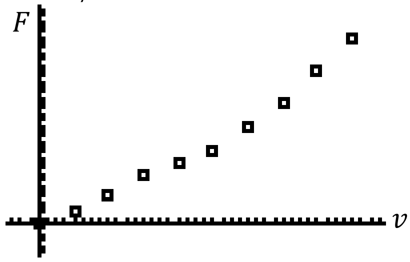
$\therefore C_1 + C_2 \sim N(43.2, 25.92)$

Required probability
$= P(C_1 + C_2 > 45) = 0.362$

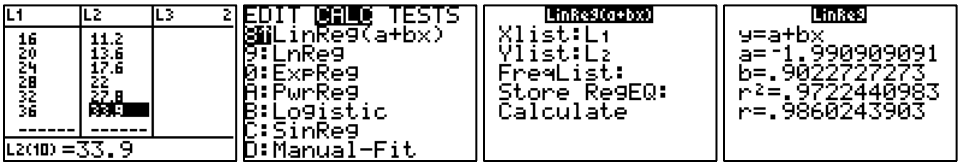
Question 10

[Ans: (i) draw (ii)(a) 0.9860 (b) 0.9907 (iii) $F = c + dv^2$ is the better model; explain
(iv) $F = 3.1957 + 0.024242v^2$; 30.7; explain]

(i) From GC,

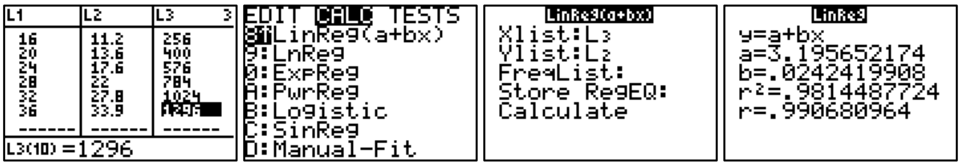


(ii) (a) From GC,



$r = 0.9860$

(b) From GC,



$r = 0.9907$

(iii) From values of r in part (ii), $F = c + dv^2$ is the better model as the value of $|r|$ is closer to 1.

(iv) From GC,

$F = 3.1957 + 0.024242v^2$

When $F = 26.0$,

$3.1957 + 0.024242v^2 = 26.0$

$v = 30.7$

As F is the dependent variable and v is the independent variable in this experiment, neither the regression lines should be used as both lines are generated assuming F to be the independent variable and v the dependent variable.

Question 11

[Ans: (i) 0.0655 (ii) 32 s (iii) 0.192 (iv) 0.235 (v) 0.959]

(i) Let X be the number of calls received in 4 minutes.

$$X \sim Po(3 \times 4) \Rightarrow X \sim Po(12)$$

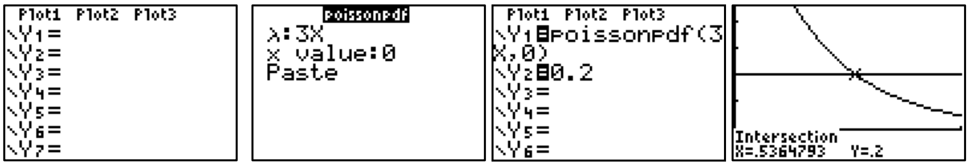
$P(X = 8) = 0.0655$

(ii) Let n be the length of time for which there is no calls, and Y be the number of calls received in n minutes.

$$Y \sim Po(3n)$$

$P(Y = 0) = 0.2$

From GC,



$n = 0.53648$ min $= 32$ s

(iii) Let A be the number of calls received in 12 hours.

$$A \sim Po(3 \times 60 \times 12)$$
$$\Rightarrow A \sim Po(2160)$$

$\lambda = 2160 > 10$

$$\therefore A \sim N(2160, 2160) \text{ approx.}$$

$P(A > 2200) = P(A \geq 2201)$
$= P(A > 2200.5)$ (c.c)
$= 0.192$

(iv) Let C be the number of working days out of 6 that are busy.

$$C \sim B(6, 0.19176)$$

$P(C = 2) = 0.235$

(v) Let D be the number of working days out of 30 that are busy.

$$D \sim B(30, 0.19176)$$

$n = 30$ (large); $np = 5.75 > 5$; $n(1 - p) = 24.2 > 5$

$$E(D) = np = 5.75, \qquad Var(D) = np(1 - p) = 4.6496$$

$$\therefore D \sim N(5.75, 4.6496) \text{ approx.}$$

$P(D < 10) = P(X \leq 9)$
$= (X < 9.5)$ (c.c)
$= 0.959$