A-LEVEL H2 MATHS 2010 – PAPER 1

Question 1

[Ans: (i) $p = \frac{3}{7}$ (ii) show]

(i) $\boldsymbol{a} = \begin{pmatrix} 2p \\ 3p \\ 6p \end{pmatrix}$; $\boldsymbol{b} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

Given $|\boldsymbol{a}| = |\boldsymbol{b}|$

$\therefore \left|\begin{pmatrix} 2p \\ 3p \\ 6p \end{pmatrix}\right| = \left|\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}\right|$

$\sqrt{(2p)^2 + (3p)^2 + (6p)^2} = \sqrt{1^2 + 2^2 + 2^2}$

$\sqrt{49p^2} = \sqrt{9}$

Since $p > 0$, $7p = 3 \Rightarrow p = \frac{3}{7}$

(ii) $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b})$

$= \begin{pmatrix} 2p+1 \\ 3p-2 \\ 6p+2 \end{pmatrix} \cdot \begin{pmatrix} 2p-1 \\ 3p+2 \\ 6p-2 \end{pmatrix}$

$= ((2p)^2 - 1^2) + ((3p)^2 - 2^2) + ((6p)^2 - 2^2)$

$= 4p^2 - 1 + 9p^2 - 4 + 36p^2 - 4$

$= 49p^2 - 9$

$= 49\left(\frac{3}{7}\right)^2 - 9$

$= 9 - 9 = 0$ (shown)

Question 2

[Ans: (i) $1+3x+\frac{5}{2}x^2+\cdots$ (ii) $n=\frac{9}{4}$; show]

(i) $e^x(1+\sin 2x)$

$$=\left(1+x+\frac{x^2}{2!}+\cdots\right)\left(1+2x-\frac{(2x)^3}{3!}+\cdots\right)$$

$$=1+2x+x+2x^2+\frac{x^2}{2}+\cdots$$

$$=1+3x+\frac{5}{2}x^2+\cdots$$

(ii) $\left(1+\frac{4}{3}x\right)^n$

$$=1+n\left(\frac{4}{3}x\right)+\frac{n(n-1)}{2!}\left(\frac{4}{3}x\right)^2+\cdots$$

$$=1+\frac{4}{3}nx+\frac{8n(n-1)}{9}x^2$$

$$\frac{4}{3}n=3\Rightarrow n=\frac{9}{4}$$

$\therefore$ 3rd term of $\left(1+\frac{4}{3}x\right)^n$

$$=\frac{16\left(\frac{9}{4}\right)\left(\frac{9}{4}-1\right)}{9}x^2$$

$$=\frac{5}{2}x^2$$

$\therefore$ the third term in each of the series are equal.

Question 3

[Ans: (i) $4n+c-2$ (ii) u_n+4]

(i) $u_n=Sn-S_{n-1}$

$=n(2n+c)-(n-1)(2(n-1)+c)$

$=2n^2+cn-(2n^2-2n+cn-2n+2+c)$

$=4n+c-2$

(ii) u_{n+1}

$=4(n+1)+c-2$

$=4n+c-2+4$

$=u_n+4$

Question 4

[Ans: (i) $\frac{dy}{dx}=\frac{x+y}{y-x}$ (ii) $\left(\sqrt{2},-\sqrt{2}\right)$, $\left(-\sqrt{2},\sqrt{2}\right)$]

(i) $x^2-y^2+2xy+4=0$

$$2x-2y\frac{dy}{dx}+2x\frac{dy}{dx}+2y=0$$

$$(2y-2x)\frac{dy}{dx}=2x+2y$$

$$\Rightarrow \frac{dy}{dx}=\frac{x+y}{y-x}$$

(ii) When tangent is // to x-axis, $\frac{dy}{dx}=0$.

$$\frac{dy}{dx}=0$$

$$\therefore \frac{x+y}{y-x}=0$$

$$\Rightarrow x=-y$$

$$\therefore (-y)^2-y^2+2(-y)y+4=0$$

$$y^2-y^2-2y^2+4=0$$

$$y=\pm\sqrt{2}$$

$$\Rightarrow x=\mp\sqrt{2}$$

$\therefore$ The coordinates are $\left(\sqrt{2},-\sqrt{2}\right)$ and $\left(-\sqrt{2},\sqrt{2}\right)$.

Question 5

[Ans: (i) $f(x)=\frac{1}{2}(x-2)^3-6$; $\left(\sqrt[3]{12}+2, 0\right)$, $(0, -10)$; sketch (ii) sketch]

(i) $y = x^3$

Translate by 2 units in positive x-direction:

$y = (x-2)^3$

Stretch with scale factor $\frac{1}{2}$ parallel to y-axis:

$y = \frac{1}{2}(x-2)^3$

Translate by 6 units in negative y-direction:

$y = \frac{1}{2}(x-2)^3 - 6 = f(x)$

When $y = 0$,

$\frac{1}{2}(x-2)^3 - 6 = 0$

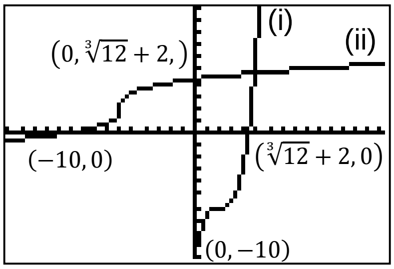
$(x-2)^3 = 12$

$x = \sqrt[3]{12} + 2$

When $x = 0$,

$y = \frac{1}{2}(-2)^3 - 6 = -10$

$\therefore$ The curve crosses the x-axis at $\left(\sqrt[3]{12}+2, 0\right)$ and the y-axis at $(0, -10)$.



Question 6

[Ans: (i) $\beta = 0.347$, $\gamma = 1.532$ (ii) 0.781 units^2 (iii) $\dfrac{9}{4}$ units^2 (iv) $\{k \in \mathbb{R}: -1 < k < 3\}$]

(i) From GC,

$\beta = 0.347$, $\gamma = 1.532$

(ii) Area

$= \left| \int_{0.347}^{1.532} x^3 - 3x + 1 \, dx \right| = 0.781$

(iii) Let $x^3 - 3x + 1 = 1$

$x^3 - 3x = 0$

$x(x^2 - 3) = 0$

$x = 0$ or $x = \pm\sqrt{3}$

Area

$= \int_{-\sqrt{3}}^{0} (x^3 - 3x + 1) - 1 \, dx$

$= \int_{-\sqrt{3}}^{0} x^3 - 3x \, dx$

$= \left[\frac{1}{4}x^4 - \frac{3}{2}x^2 \right]_{-\sqrt{3}}^{0}$

$= 0 - \left[\frac{1}{4}\left(\sqrt{3}\right)^4 - \frac{3}{2}\left(\sqrt{3}\right)^2 \right] = \frac{9}{4}$

(iv) When $x = -1$, $y = 3$

When $x = 1$, $y = -1$

$\therefore$ From observation, $\{k \in \mathbb{R}: -1 < k < 3\}$

Question 7

[Ans: show; $10\ln 2$; θ tends to 20; sketch]

Let $\frac{d\theta}{dt} = k(20-\theta)$, $k > 0$

When $t = 0$, $\theta = 10$ & $\frac{d\theta}{dt} = 1$

$\therefore 1 = k(20-10) \Rightarrow k = \frac{1}{10}$

$\therefore \frac{d\theta}{dt} = \frac{1}{10}(20-\theta)$

$\frac{1}{20-\theta}\frac{d\theta}{dt} = \frac{1}{10}$

$\int \frac{1}{20-\theta}\,d\theta = \int \frac{1}{10}\,dt$

$-\ln|20-\theta| = \frac{1}{10}t + A$

$\ln|20-\theta| = -\frac{1}{10}t - A$

$20-\theta = \pm e^{-\frac{1}{10}t-A} = Be^{-\frac{1}{10}t}$

$\theta = 20 - Be^{-\frac{1}{10}t}$

When $t = 0$,

$\theta = 10$

$20 - B = 10 \Rightarrow B = 10$

$\therefore \theta = 20 - 10e^{-\frac{1}{10}t}$ (shown)

When $\theta = 15$,

$20 - 10e^{-\frac{1}{10}t} = 15$

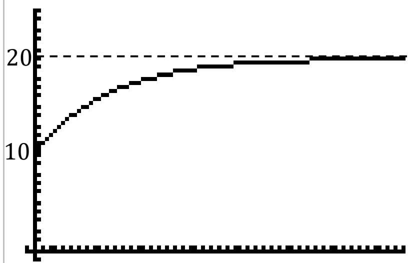
$10e^{-\frac{1}{10}t} = 5$

$e^{-\frac{1}{10}t} = \frac{1}{2}$

$-\frac{1}{10}t = \ln\frac{1}{2} = -\ln 2$

$t = 10\ln 2$

When $t \to \infty$, $\theta \to 20$



Question 8

[Ans: (i) $z_1 = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$, $z_2 = 2\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$

(ii) $\sqrt{2}\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right]$ (iii) sketch (iv) 2]

(i) $|z_1| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$

$\arg z_1 = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$

$\therefore z_1 = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$

$|z_2| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\arg z_2 = -\pi + \tan^{-1} 1 = -\frac{3\pi}{4}$

$\therefore z_2 = \sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$

(ii) $\frac{z_1}{z_2}$

$= \frac{2}{\sqrt{2}}\left[\cos\left(\frac{\pi}{3} - \left(-\frac{3\pi}{4}\right)\right) + i\sin\left(\frac{\pi}{3} - \left(-\frac{3\pi}{4}\right)\right)\right]$

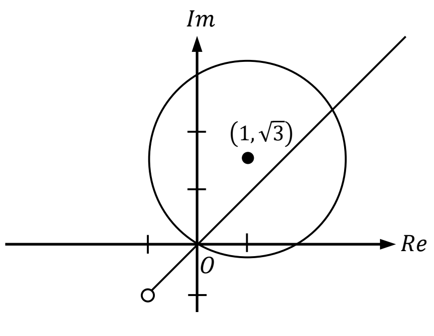
$= \sqrt{2}\left[\cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)\right]$

$= \sqrt{2}\left[\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right]$

$\therefore \left(\frac{z_1}{z_2}\right)^* = \sqrt{2}\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right]$

(iii) $|z - z_1| = 2 \Rightarrow \left|z - \left(1 + i\sqrt{3}\right)\right| = 2$

$\arg(z - z_2) = \frac{\pi}{4} \Rightarrow \arg\left(z - (-1 - i)\right) = \frac{\pi}{4}$



(iv) From observation, intersection with positive real axis is at 2.

Question 9

[Ans: (i) $x=\sqrt[3]{\frac{200(1+k)}{3}}$ (ii) $\frac{3}{2(1+k)}$ (iii) $\frac{3}{4}\le\frac{y}{x}<\frac{3}{2}$ (iv) $\frac{1}{2}$]

(i) $(x)(3x)(y)=300$

$\Rightarrow y=\frac{100}{x^2}$

External surface area, A

$= 2(xy)+2(3xy)+(x)(3x)+2(kxy)+2(3kxy)+(x)(3x)$

$= 8xy+8kxy+6x^2$

$= 8(1+k)xy+6x^2$

$= 8(1+k)x\left(\frac{100}{x^2}\right)+6x^2$

$= \frac{800(1+k)}{x}+6x^2$

$\frac{dA}{dx}=-\frac{800(1+k)}{x^2}+12x$

Let $\frac{dA}{dx}=0$,

$-\frac{800(1+k)}{x^2}+12x=0$

$12x^3=800(1+k)$

$x^3=\frac{200(1+k)}{3}\Rightarrow x=\sqrt[3]{\frac{200(1+k)}{3}}$

$\frac{d^2A}{dx^2}=\frac{1600(1+k)}{x^3}+12$

Since $x>0$

$\therefore \frac{d^2A}{dx^2}>0$

$\therefore\ x=\sqrt[3]{\frac{200(1+k)}{3}}$ gives the minimum external surface area.

(ii) When $x=\sqrt[3]{\frac{200(1+k)}{3}}$,

$y=\frac{100}{x^2}$

$\frac{y}{x}=\frac{100}{x^3}=\frac{100}{\frac{200(1+k)}{3}}$

$\frac{y}{x}=\frac{3}{2(1+k)}$

(iii) $0<k\le 1\Rightarrow 2<2(1+k)\le 4$

$\frac{1}{2}>\frac{1}{2(1+k)}\ge\frac{1}{4}$

$\frac{3}{4}\le\frac{3}{2(1+k)}<\frac{3}{2}\Rightarrow\frac{3}{4}\le\frac{y}{x}<\frac{3}{2}$

(iv) For square ends,

$$\frac{y}{x} = 1 \Rightarrow \frac{3}{2(1+k)} = 1$$

$$1 + k = \frac{3}{2} \Rightarrow k = \frac{1}{2}$$

Question 10

[Ans: (i) show (ii) $\left(\frac{17}{2}, 2, \frac{3}{2}\right)$ (iii) show; $B(19, -19, -30)$ (iv) 348 units^2]

$l: \dfrac{x-10}{-3} = \dfrac{y+1}{6} = \dfrac{z+3}{9}$

$\Rightarrow l: \mathbf{r} = \begin{pmatrix} 10 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 6 \\ 9 \end{pmatrix}$

$\Rightarrow l: \mathbf{r} = \begin{pmatrix} 10 \\ -1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$

$p: x - 2y - 3z = 0 \Rightarrow p: \mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 0$

(i) Direction of $l = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = -\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$

∴ l is parallel to normal of p

⇒ l is perpendicular to p. (shown)

(ii) $\begin{pmatrix} 10-\mu \\ -1+2\mu \\ -3+3\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 0$

$10 - \mu + 2 - 4\mu + 9 - 9\mu = 0$

$14\mu = 21 \Rightarrow \mu = \frac{3}{2}$

∴ Point of intersection

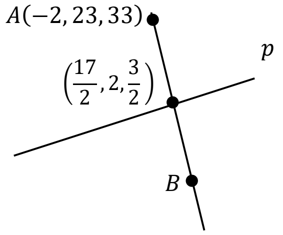
$= \begin{pmatrix} 10 - 3/2 \\ -1+3 \\ -3 + 9/2 \end{pmatrix} = \begin{pmatrix} 17/2 \\ 2 \\ 3/2 \end{pmatrix}$

∴ Point is intersection is at $\left(\frac{17}{2}, 2, \frac{3}{2}\right)$.

(iii) Let $\begin{pmatrix} 10-\mu \\ -1+2\mu \\ -3+3\mu \end{pmatrix} = \begin{pmatrix} -2 \\ 23 \\ 33 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \mu \\ \mu \\ \mu \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 12 \end{pmatrix}$

∴ $(-2, 23, 33)$ lies on l.



$$\begin{pmatrix} 17/2 \\ 2 \\ 3/2 \end{pmatrix} = \frac{\begin{pmatrix} -2 \\ 23 \\ 33 \end{pmatrix} + \overrightarrow{OB}}{1+1}$$

$$\overrightarrow{OB} = 2\begin{pmatrix} 17/2 \\ 2 \\ 3/2 \end{pmatrix} - \begin{pmatrix} -2 \\ 23 \\ 33 \end{pmatrix} = \begin{pmatrix} 19 \\ -19 \\ -30 \end{pmatrix}$$

$\therefore B(19,-19,-30)$

(iv) Area

$$= \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right|$$

$$= \frac{1}{2}\left|\begin{pmatrix} -2 \\ 23 \\ 33 \end{pmatrix} \times \begin{pmatrix} 19 \\ -19 \\ -30 \end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix} -63 \\ 567 \\ -399 \end{pmatrix}\right|$$

$= 348$ units^2

Question 11

[Ans: (i) show (ii) show (iii) $x^2 - y^2 = 4$; sketch]

(i) $\frac{dx}{dt} = 1 - \frac{1}{t^2}$; $\frac{dy}{dt} = 1 + \frac{1}{t^2}$

$$\therefore \frac{dy}{dx} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \frac{t^2 + 1}{t^2 - 1}$$

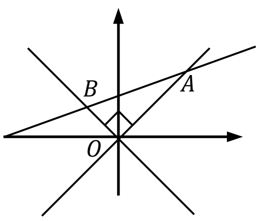
At point P,

$$x = p + \frac{1}{p} = \frac{p^2 + 1}{p}$$
$$y = p - \frac{1}{p} = \frac{p^2 - 1}{p}$$
$$\frac{dy}{dx} = \frac{p^2 + 1}{p^2 - 1}$$

Equation of tangent:

$$y - \frac{p^2 - 1}{p} = \frac{p^2 + 1}{p^2 - 1}\left(x - \frac{p^2 + 1}{p}\right)$$
$$\frac{py - (p^2 - 1)}{p} = \frac{p^2 + 1}{p^2 - 1}\left(\frac{px - (p^2 + 1)}{p}\right)$$
$$py - (p^2 - 1) = \frac{p^2 + 1}{p^2 - 1}\left(px - (p^2 + 1)\right)$$
$$p(p^2 - 1)y - (p^2 - 1)^2 = p(p^2 + 1)x - (p^2 + 1)^2$$
$$p(p^2 + 1)x - p(p^2 - 1)y = (p^2 + 1)^2 - (p^2 - 1)^2$$
$$p(p^2 + 1)x - p(p^2 - 1)y = (p^2 + 1 + p^2 - 1)(p^2 + 1 - p^2 + 1)$$
$$p(p^2 + 1)x - p(p^2 - 1)y = (2p^2)(2)$$
$$(p^2 + 1)x - (p^2 - 1)y = 4p \text{ (shown)}$$

(ii)



At A,

$$(p^2 + 1)x - (p^2 - 1)x = 4p$$
$$2x = 4p \Rightarrow x = 2p$$
$$\therefore A(2p, 2p)$$

At B,

$$(p^2 + 1)x - (p^2 - 1)(-x) = 4p$$
$$2p^2x = 4p \Rightarrow x = \frac{2}{p}$$
$$\therefore B\left(\frac{2}{p}, -\frac{2}{p}\right)$$

Area

$$= \frac{1}{2}\sqrt{(2p)^2 + (2p)^2}\sqrt{\left(\frac{2}{p}\right)^2 + \left(\frac{2}{p}\right)^2}$$

$$= \frac{1}{2}\sqrt{(8p^2)\left(\frac{8}{p^2}\right)}$$

$= 4$ units^2

$\therefore$ The area is independent of p. (shown)

(iii) $x + y = \left(t + \frac{1}{t}\right) + \left(t - \frac{1}{t}\right)$

$$x + y = 2t \Rightarrow t = \frac{x+y}{2}$$

$$y = t - \frac{1}{t}$$

$$y = \frac{x+y}{2} - \frac{2}{x+y}$$

$$2y(x+y) = (x+y)^2 - 4$$

$$2xy + 2y^2 = x^2 + 2xy + y^2 - 4$$

$$x^2 - y^2 = 4$$

$$\Rightarrow \frac{x^2}{2^2} - \frac{y^2}{2^2} = 1$$