A-LEVEL H2 MATH 2025 – PAPER 1

Question 1

[Ans: (a) $u_2 = 3a + 5$, $u_3 = 9a + 20$ (b) $a = \frac{1}{2}$]

(a) $u_2 = 3u_1 + 5 = 3a + 5$

$u_3 = 3u_2 + 5 = 3(3a+5) + 5 = 9a + 20$

(b) $u_4 = 3u_3 + 5 = 3(9a+20) + 5 = 27a + 65$

$\sum_{r=1}^{4} u_r = 110$

$u_1 + u_2 + u_3 + u_4 = 110$

$(a) + (3a+5) + (9a+20) + (27a+65) = 110$

$40a = 20 \Rightarrow a = \frac{1}{2}$

Question 2

[Ans: (a) $a = 8$, $d = -\frac{1}{2}$ (b) show]

(a) $S_{20} = 65$

$\frac{20}{2}\left[2a + (20-1)d\right] = 65$

$4a + 38d = 13 - - - (1)$

$S_{28} - S_{20} = -30$

$S_{28} = S_{20} - 30$

$= 65 - 30 = 35$

$\frac{28}{2}\left[2a + (28-1)d\right] = 35$

$28a + 378d = 35 - - - (2)$

Solving (1) and (2): $a = 8, d = -\frac{1}{2}$

(b) $\frac{u_{n+1}}{u_n} = \frac{e^{k(n+2)}}{e^{k(n+1)}}$

$= e^{k(n+2) - k(n+1)}$

$= e^k$ (constant)

$\therefore$ The sequence is geometric with common ratio of e^k.

Question 3

[Ans: (a) $|z|=1$, $\arg z=\frac{\pi}{3}$ (b) $B: w=iz$]

(a) $z=\frac{\sqrt{3}+i}{\sqrt{3}-i}$

$=\frac{\sqrt{3}+i}{\sqrt{3}-i}\times\frac{\sqrt{3}+i}{\sqrt{3}+i}$

$=\frac{3+2\sqrt{3}i-1}{\left(\sqrt{3}\right)^2+1^2}=\frac{2+2\sqrt{3}i}{4}$

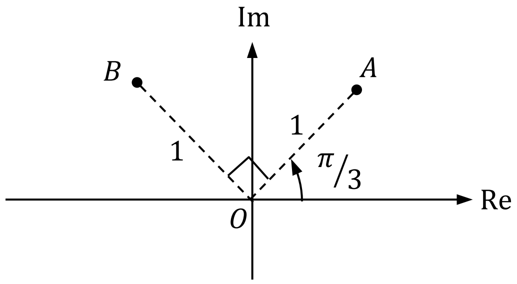
$=\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$|z|=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=1$

Basic angle $=\tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)=\frac{\pi}{3}$

$\arg(z)=\frac{\pi}{3}$ (1st Quadrant)

(b) Let w be the complex number representing point B.



$w=iz$.

As point B is obtained by the rotation of point A through $\frac{\pi}{2}$ in an anti-clockwise direction about the origin O with $OA=OB$, the complex number w will be derived by mulitplying z by i.

Question 4

[Ans: (a) $\frac{dy}{dx}=\frac{3x-4}{2(x+1)^{\frac{3}{2}}}$, $x=\frac{4}{3}$ (b) $\frac{d^2y}{dx^2}=0.421$; minimum point]

(a) $y=\frac{3x+10}{\sqrt{x+1}}$

$$\frac{dy}{dx}=\frac{\sqrt{x+1}(3)-(3x+10)\frac{1}{2}(x+1)^{-\frac{1}{2}}}{x+1}$$

$$=\frac{3\sqrt{x+1}-\frac{3x+10}{2\sqrt{x+1}}}{x+1}$$

$$=\frac{6(x+1)-(3x+10)}{2(x+1)^{\frac{3}{2}}}$$

$$=\frac{3x-4}{2(x+1)^{\frac{3}{2}}}$$

For turning point, let $\frac{dy}{dx}=0$,

$$\frac{3x-4}{2(x+1)^{\frac{3}{2}}}=0$$

$x=\frac{4}{3}$ (x- coordinate of turning point)

(b) $\frac{dy}{dx}=\frac{3x-4}{2(x+1)^{\frac{3}{2}}}$

$$\frac{d^2y}{dx^2}=\frac{2(x+1)^{\frac{3}{2}}(3)-(3x-4)\frac{3}{2}\left[2(x+1)^{\frac{1}{2}}\right]}{4(x+1)^3}$$

$$=\frac{6(x+1)^{\frac{3}{2}}-3(3x-4)(x+1)^{\frac{1}{2}}}{4(x+1)^3}$$

$$=\frac{3(x+1)^{\frac{1}{2}}\left[2(x+1)-(3x-4)\right]}{4(x+1)^3}=\frac{3(x+1)^{\frac{1}{2}}(6-x)}{4(x+1)^3}$$

When $x=\frac{4}{3}$,

$$\frac{d^2y}{dx^2}=0.421>0$$

$\therefore$ The is a minimum point.

Question 5

[Ans: (a) $x \geq \ln 2$ (b) $2\ln 2 - 4 + e + \frac{2}{e}$]

(a) $e^x \geq 2e^{-x} + 1$

$e^x - 2e^{-x} - 1 \geq 0$

$e^{2x} - e^x - 2 \geq 0$ ($\because e^x > 0$)

$\left(e^x - 2\right)\left(e^x + 1\right) \geq 0$

$e^x \leq -1$ (NA) or $e^x \geq 2$

$\therefore x \geq \ln 2$

(b) $\int_0^1 \left|e^x - 2e^{-x} - 1\right| dx$

$= \int_0^{\ln 2} -\left(e^x - 2e^{-x} - 1\right) dx + \int_{\ln 2}^1 e^x - 2e^{-x} - 1 dx$

$= -\left[e^x + 2e^{-x} - x\right]_0^{\ln 2} + \left[e^x + 2e^{-x} - x\right]_{\ln 2}^1$

$= -\left[e^{\ln 2} + 2e^{-\ln 2} - \ln 2 - \left(1 + 2 - 0\right)\right] + \left[e + 2e^{-1} - 1 - \left(e^{\ln 2} + 2e^{-\ln 2} - \ln 2\right)\right]$

$= -\left(2 + \frac{2}{2} - \ln 2 - 3\right) + \left(e + \frac{2}{e} - 1 - 2 - \frac{2}{2} + \ln 2\right)$

$= 2\ln 2 - 4 + e + \frac{2}{e}$

Question 6

[Ans: (a) $k=\frac{1}{5730}\ln 2$ (b) 13300 years]

(a) $\frac{dM}{dt}=-kM$

$\int \frac{1}{M}\,dM=\int -k\,dt$

$\ln|M|=-kt+C$

$M=\pm e^{C}e^{-kt}$

$M=Ae^{-kt}$, where $A=\pm e^{C}$

When $t=0$, $M=M_0$

$\Rightarrow A=M_0$

$\therefore M=M_0e^{-kt}$

When $t=5730$, $M=\frac{1}{2}M_0$

$\frac{1}{2}M_0=M_0e^{-k(5730)}$

$e^{-k(5730)}=\frac{1}{2}$

$-5730k=\ln\left(\frac{1}{2}\right)$

$k=-\frac{1}{5730}\ln\left(\frac{1}{2}\right)$

$k=\frac{1}{5730}\ln 2$

(b) Let $M=0.2M_0$

$0.2M_0=M_0e^{-\left(\frac{1}{5730}\ln 2\right)t}$

$0.2=e^{-\left(\frac{1}{5730}\ln 2\right)t}$

$-\left(\frac{1}{5730}\ln 2\right)t=\ln 0.2$

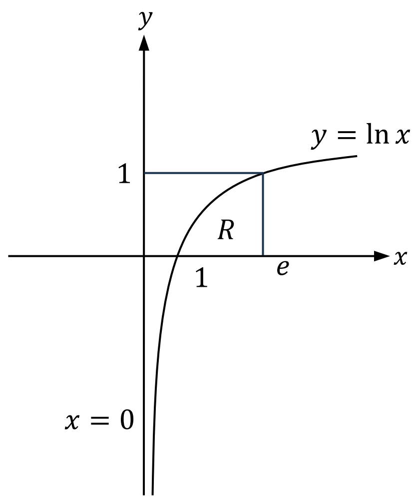
$t=13300$ years (to the nearest 100 years)

Question 7

[Ans: (a) show (b) $\pi\left(e-2\right)$ units² (c) 13.18 units³]

(a) $\dfrac{d}{dx}\left(x\ln x-x\right)=x\left(\dfrac{1}{x}\right)+\ln x\left(1\right)-1=\ln x$ (shown)

(b) Volume (x - axis)



$=\pi\int_1^e y^2 dx$

$=\pi\int_1^e \left(\ln x\right)^2 dx$

Let $u=\left(\ln x\right)^2$, $\dfrac{dv}{dx}=1$

$\dfrac{du}{dx}=2\left(\ln x\right)\dfrac{1}{x}$, $v=x$

$=\pi\left\{\left[x\left(\ln x\right)^2\right]_1^e-\int_1^e x\left[2\left(\ln x\right)\dfrac{1}{x}\right] dx\right\}$

$=\pi\left\{e\left(\ln e\right)^2-\left(0\right)-2\int_1^e \ln x\ dx\right\}$

$=\pi\left\{e-2\left[x\ln x-x\right]_1^e\right\}$

$=\pi\left\{e-2\left[\left(e\ln e-e\right)-\left(0-1\right)\right]\right\}$

$=\pi\left(e-2\right)$

(c) Volume (y - axis)

$=\pi\left(e\right)^2\left(1\right)-\pi\int_0^1 x^2 dy$

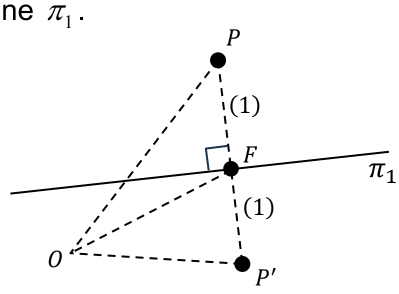
$=\pi e^2-\pi\int_0^1 e^{2y} dy$

$=13.18$

Question 8

[Ans: (a) $\begin{pmatrix} -2 \\ 3/2 \\ 3/2 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ (c) $85.9°$]

(a) Let F be the foot of perpendicular from P to the plane π_1.



$\pi_1 : \mathbf{r}.\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 2$

$l_{PF} : \mathbf{r} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3+2\lambda \\ 1+\lambda \\ 3\lambda \end{pmatrix}$

Since F lies on both π_1 and l_{PF}, $\begin{pmatrix} -3+2\lambda \\ 1+\lambda \\ 3\lambda \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 2$

$2(-3+2\lambda)+1(1+\lambda)+3(3\lambda) = 2 \Rightarrow \lambda = \frac{1}{2}$

$\therefore \overrightarrow{OF} = \begin{pmatrix} -3+2\left(1/2\right) \\ 1+\left(1/2\right) \\ 3\left(1/2\right) \end{pmatrix} = \begin{pmatrix} -2 \\ 3/2 \\ 3/2 \end{pmatrix}$

(b) Let P' be the required reflected point.

By ratio theorem, $\overrightarrow{OF} = \frac{(1)\overrightarrow{OP} + (1)\overrightarrow{OP'}}{1+1} = \frac{1}{2}\left(\overrightarrow{OP} + \overrightarrow{OP'}\right)$

$\overrightarrow{OP'} = 2\overrightarrow{OF} - \overrightarrow{OP} = 2\begin{pmatrix} -2 \\ 3/2 \\ 3/2 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$

(c) Normal of $\pi_2 = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -13 \\ 4 \\ 6 \end{pmatrix}$

Let θ be the required angle between π_1 and π_2.

$$\cos\theta = \left| \frac{\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -13 \\ 4 \\ 6 \end{pmatrix}}{\sqrt{2^2+1^2+3^2}\sqrt{13^2+4^2+6^2}} \right| = \left| \frac{4}{\sqrt{14}\sqrt{221}} \right| \Rightarrow \theta = 85.9°$$

Question 9

[Ans: (a) show (b) $\sin^{-1} px = px + \frac{p^3}{6}x^3 + ...$ (c) $\frac{1}{\sqrt{1-9x^2}} = 1 + \frac{9}{2}x^2 + ...$]

(a) $y = \sin^{-1} px$

$\sin y = px$

$\cos y \frac{dy}{dx} = p$

$\frac{dy}{dx} = p \sec y$ (shown)

(b) $\frac{d^2 y}{dx^2} = p \sec y \tan y \frac{dy}{dx} \Rightarrow \frac{d^2 y}{dx^2} = p^2 \sec^2 y \tan y$

$$\frac{d^3 y}{dx^3} = p^2 \left[\sec^2 y \left(\sec^2 y \frac{dy}{dx} \right) + \tan y \left(2 \sec y \sec y \tan y \right) \frac{dy}{dx} \right]$$

$$\frac{d^3 y}{dx^3} = p^2 \left(\sec^4 y \frac{dy}{dx} + 2 \sec^2 y \tan^2 y \frac{dy}{dx} \right)$$

When $x = 0$,

$y = 0$, $\frac{dy}{dx} = p$, $\frac{d^2 y}{dx^2} = 0$, $\frac{d^3 y}{dx^3} = p^2 \left[(1)(p) + 2(1)(0)(p) \right] = p^3$

$$\therefore \sin^{-1} px = 0 + x(p) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}\left(p^3\right) + ...$$

$$\sin^{-1} px = px + \frac{p^3}{6}x^3 + ...$$

(c) $y = \sin^{-1} px = px + \frac{p^3}{6}x^3 + ...$

$$\frac{dy}{dx} = \frac{p}{\sqrt{1-(px)^2}} = p + \frac{p^3}{2}x^2 + ...$$

Let $p = 3$,

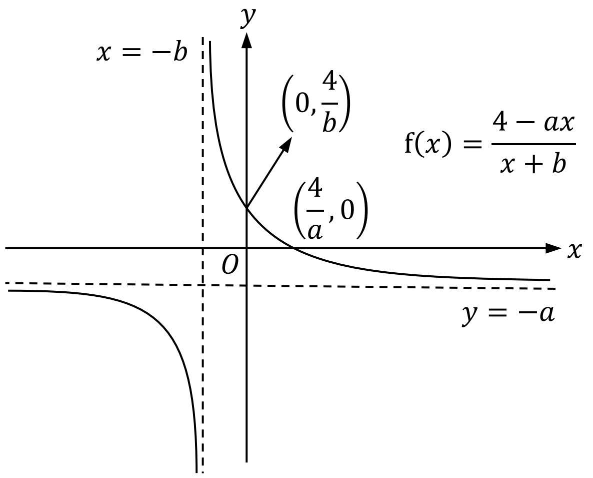
$$\frac{3}{\sqrt{1-(3x)^2}} = 3 + \frac{3^3}{2}x^2 + ...$$

$$\therefore \frac{1}{\sqrt{1-9x^2}} = 1 + \frac{9}{2}x^2 + ...$$

Question 10

[Ans: (a) sketch (b) describe (c) $\mathrm{f}^{-1}(x)=\dfrac{4-bx}{x+a}$, $R_{\mathrm{f}^{-1}}=\mathbb{R}\setminus\{-b\}$ (d) $\mathrm{f}^{2025}(x)=\dfrac{4-2a}{2+a}$]

(a) $\mathrm{f}(x)=\dfrac{4-ax}{x+b}=-a+\dfrac{4+ab}{x+b}$



(b) Let $y=\dfrac{4-ax}{x+b}=-a+\dfrac{4+ab}{x+b}\to(1):y=\dfrac{4+ab}{x+b}\to(2):y=\dfrac{1}{x+b}\to(3):y=\dfrac{1}{x}$

(1): Translate a units in the positive direction of y.

(2): Scaling parallel to the y - axis by a factor of $\dfrac{1}{4+ab}$.

(3): Translate b units in the positive direction of x.

(c) $y=\dfrac{4-ax}{x+b}$

$xy+by=4-ax$

$xy+ax=4-by\Rightarrow x=\dfrac{4-by}{y+a}$

$\therefore \mathrm{f}^{-1}(x)=\dfrac{4-bx}{x+a}$

$R_{\mathrm{f}^{-1}}=D_{\mathrm{f}}=\mathbb{R}\setminus\{-b\}$

(d) When $b=a\Rightarrow \mathrm{f}(x)=\mathrm{f}^{-1}(x)=\dfrac{4-ax}{x+a}\Rightarrow \mathrm{f}$ is a self-inverse function.

$\mathrm{f}^2(x)=\mathrm{ff}(x)=\mathrm{ff}^{-1}(x)=x$

$\therefore \mathrm{f}^2(x)=\mathrm{f}^4(x)=\mathrm{f}^6(x)=...=\mathrm{f}^{2024}(x)=x$

$\mathrm{f}^{2025}(x)=\mathrm{ff}^{2024}(x)=\mathrm{f}(x)$

$\mathrm{f}(2)=\dfrac{4-a(2)}{2+a}=\dfrac{4-2a}{2+a}$

Question 11

[Ans: (a) $\frac{dy}{dx}=\frac{5(\theta\cos\theta+\sin\theta)}{7(\theta\sin\theta-\cos\theta)}$ (b)(i) $\theta=-0.86$, 0.86 (ii) $CD=31.4$

(c)(i) $\theta=0$ (ii) $\theta=\pm\frac{\pi}{2}$ (iii) $AB=31.4$ (d) no]

(a) $x=28(\theta\cos\theta+1)\Rightarrow\frac{dx}{d\theta}=28(-\theta\sin\theta+\cos\theta)$

$y=20(2-\theta\sin\theta)\Rightarrow\frac{dy}{d\theta}=20(-\theta\cos\theta-\sin\theta)$

$$\frac{dy}{dx}=\frac{20(-\theta\cos\theta-\sin\theta)}{28(-\theta\sin\theta+\cos\theta)}=\frac{5(\theta\cos\theta+\sin\theta)}{7(\theta\sin\theta-\cos\theta)}$$

(b) (i) At point C and D,

$\frac{dy}{dx}$ is undefined $\Rightarrow\theta\sin\theta-\cos\theta=0$

Using GC, $\theta=-0.86033$, 0.86033

$\therefore\theta=-0.86$, $\theta=0.86$ (2 significant figures)

(ii) When $\theta=-0.86033$, $x=12.289$, $y=26.956$ (Point C)

When $\theta=0.86033$, $x=43.711$, $y=26.956$ (Point D)

$\therefore CD=43.711-12.289=31.4$

(c) (i) When $x=28$ (given),

$28=28(\theta\cos\theta+1)\Rightarrow\theta\cos\theta=0\Rightarrow\theta=0$ or $\theta=\pm\frac{\pi}{2}$

When $\theta=0$, $y=40$;

When $\theta=-\frac{\pi}{2}$, $y=8.5841$;

When $\theta=\frac{\pi}{2}$, $y=8.5841$

$\therefore\theta=0$ since it gives the highest point of the three, which will be point A.

(ii) The 2 possible values of $\theta=\pm\frac{\pi}{2}$ (from part (c)(i))

(iii) $AB=40-8.5841=31.4$

(d) y-coordinates of E is same as that of C and D, which is $y=26.956$

$AE=40-26.956=13.044$

$\frac{1}{2}CD=\frac{1}{2}(43.711-12.289)=15.711$

To comply with the safety regulation,

$0.9(15.711)<AE<1.1(15.711)\Rightarrow14.140<AE<17.282$

Since $AE=13.044$ falls outside the specified range, the design of the track does not satisfy the safety regulation.

Question 12

[Ans: (a) $1-2i$ (b) $p=-4$, $q=9$, 2 (c) $w=\frac{1}{5}-\frac{2}{5}i$, $\frac{1}{5}+\frac{2}{5}i$, $\frac{1}{2}$]

(a) Let $\mathrm{f}(z)=z^3+pz^2+qz-10$

As all the coefficients of $\mathrm{f}(z)$ are real, by conjugate root theorem, since $1+2i$ is a root of $\mathrm{f}(z)=0$, the other root will be $1-2i$.

(b) A quadratic factor $\mathrm{f}(z)$

$=\left[z-(1+2i)\right]\left[z-(1-2i)\right]=z^2-2z+5$

Let $z^3+pz^2+qz-10=\left(z^2-2z+5\right)(z-2)$

Coefficient of z^2: $p=-2-2=-4$
Coefficient of z^1: $q=4+5=9$

The third root is 2.

(c) $1+pw+qw^2-10w^3=0$

$\left(\frac{1}{w}\right)^3+p\left(\frac{1}{w}\right)^2+q\left(\frac{1}{w}\right)-10=0$ ($\because w\neq 0$)

Replacing z with $\frac{1}{w}$,

$\frac{1}{w}=1+2i$, $1-2i$, 2

Roots are:

$w=\frac{1}{1+2i}\times\frac{1-2i}{1-2i}=\frac{1}{5}-\frac{2}{5}i$

$w=\frac{1}{1-2i}\times\frac{1+2i}{1+2i}=\frac{1}{5}+\frac{2}{5}i$

$w=\frac{1}{2}$