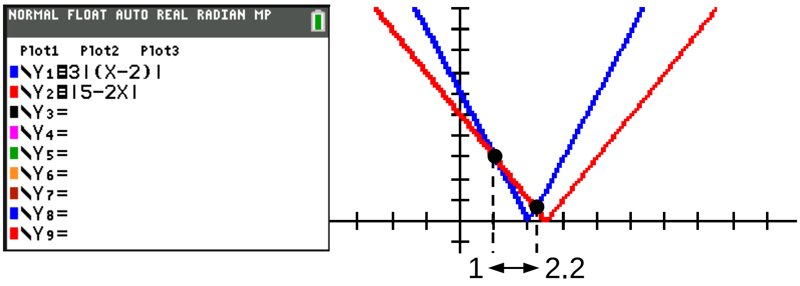
A-LEVEL H2 MATH 2023 - PAPER 2

Question 1

[Ans: (a) $1<x<2.2$ (b) $\dfrac{3x^2-11x+10}{x^2-4x-5}$; $x<-1$ or $\dfrac{5}{3}<x<2$ or $x>5$]

(a)



$\{x\in\mathbb{R}:1<x<2.2\}$

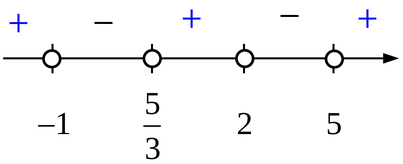
(b) $\dfrac{x+25}{x^2-4x-5}+3=\dfrac{x+25+3x^2-12x-15}{x^2-4x-5}=\dfrac{3x^2-11x+10}{x^2-4x-5}$

$$\frac{x+25}{x^2-4x-5}>-3$$

$$\frac{x+25}{x^2-4x-5}+3>0$$

$$\frac{3x^2-11x+10}{x^2-4x-5}>0$$

$$\frac{(3x-5)(x-2)}{(x+1)(x-5)}>0$$



$x<-1$ or $\dfrac{5}{3}<x<2$ or $x>5$

Question 2

[Ans: (a) show (b) $y=\frac{1}{2}x^2+\frac{1}{12}x^4+\ldots$ (c) $\ln 2\approx\frac{1}{16}\pi^2+\frac{1}{1536}\pi^4$ (d) 0.005219]

(a) $y=\ln(\sec x)$

$e^y=\sec x$

$e^y\frac{dy}{dx}=\sec x\tan x=e^y\tan x$

$\frac{dy}{dx}=\tan x$

$\frac{d^2y}{dx^2}=\sec^2 x=e^{2y}$

$\frac{d^3y}{dx^3}=e^{2y}\left(2\frac{dy}{dx}\right)=2\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)$ (shown)

(b) $\frac{d^4y}{dx^4}=2\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2y}{dx^2}\right)+2\left(\frac{d^3y}{dx^3}\right)\left(\frac{dy}{dx}\right)=2\left(\frac{d^2y}{dx^2}\right)^2+2\left(\frac{d^3y}{dx^3}\right)\left(\frac{dy}{dx}\right)$

When $x=0$,

$y=\ln(\sec 0)=0$, $\frac{dy}{dx}=\tan 0=0$, $\frac{d^2y}{dx^2}=e^0=1$,

$\frac{d^3y}{dx^3}=2(1)(0)=0$, $\frac{d^4y}{dx^4}=2(1)^2+2(0)(0)=2$

$\therefore y=0+(0)x+\frac{1}{2!}x^2+(0)x^3+\frac{2}{4!}x^4+\ldots=\frac{1}{2}x^2+\frac{1}{12}x^4+\ldots$

(c) $y=\ln(\sec x)=\frac{1}{2}x^2+\frac{1}{12}x^4+\ldots$

When $x=\frac{1}{4}\pi$,

$\ln\left(\sec\frac{1}{4}\pi\right)\approx\frac{1}{2}\left(\frac{1}{4}\pi\right)^2+\frac{1}{12}\left(\frac{1}{4}\pi\right)^4$

$\ln\left(\sqrt{2}\right)\approx\frac{1}{32}\pi^2+\frac{1}{3072}\pi^4$

$\frac{1}{2}\ln 2\approx\frac{1}{32}\pi^2+\frac{1}{3072}\pi^4$

$\ln 2\approx\frac{1}{16}\pi^2+\frac{1}{1536}\pi^4$

(d) $\int_0^{\frac{1}{10}\pi}\ln(\sec x)\,dx\approx\int_0^{\frac{1}{10}\pi}\frac{1}{2}x^2+\frac{1}{12}x^4\,dx=0.005219$

Question 3

[Ans: (a) $\begin{pmatrix} 5 \\ -10 \\ 14 \end{pmatrix}$ (b) 8.5 (c) 83.8° (d) $\begin{pmatrix} 7/72 \\ 7/36 \\ 241/36 \end{pmatrix}$]

(a) $\overrightarrow{BC} = 2\overrightarrow{AB}$

$\overrightarrow{OC} - \overrightarrow{OB} = 2\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$

$\overrightarrow{OC} = 3\overrightarrow{OB} - 2\overrightarrow{OA} = 3\begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -10 \\ 14 \end{pmatrix}$

(b) $\overrightarrow{AD} = \begin{pmatrix} 1 \\ 2 \\ d \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ d-5 \end{pmatrix}$; $\overrightarrow{BD} = \begin{pmatrix} 1 \\ 2 \\ d \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ d-8 \end{pmatrix}$

$\left|\overrightarrow{AD}\right| = \left|\overrightarrow{BD}\right|$

$\left|\begin{pmatrix} 2 \\ 0 \\ d-5 \end{pmatrix}\right| = \left|\begin{pmatrix} 0 \\ 4 \\ d-8 \end{pmatrix}\right|$

$\sqrt{2^2 + 0^2 + (d-5)^2} = \sqrt{0^2 + 4^2 + (d-8)^2}$

$d^2 - 10d + 29 = d^2 - 16d + 80$

$6d = 51 \Rightarrow d = 8.5$

(c) $\overrightarrow{AD} = \begin{pmatrix} 2 \\ 0 \\ 3.5 \end{pmatrix}$; $\overrightarrow{BD} = \begin{pmatrix} 0 \\ 4 \\ 0.5 \end{pmatrix}$

$\left|\overrightarrow{AD}\right| = \left|\overrightarrow{BD}\right| = \sqrt{2^2 + 0^2 + 3.5^2} = \sqrt{16.25}$

$\overrightarrow{AD} \bullet \overrightarrow{BD} = \left|\overrightarrow{AD}\right|\left|\overrightarrow{BD}\right| \cos \angle ADB$

$$\cos \angle ADB = \frac{\overrightarrow{AD} \bullet \overrightarrow{BD}}{\left|\overrightarrow{AD}\right|\left|\overrightarrow{BD}\right|} = \frac{\begin{pmatrix} 2 \\ 0 \\ 3.5 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 4 \\ 0.5 \end{pmatrix}}{\sqrt{16.25}\sqrt{16.25}} = \frac{1.75}{16.25} = \frac{7}{65}$$

$\angle ADB = \cos^{-1}\left(\frac{7}{65}\right) = 83.8°$

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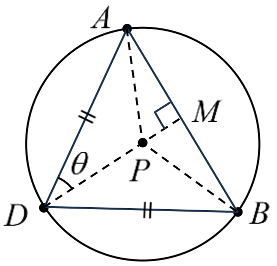
(d)



$$\angle APB = 2\angle ADB$$
$$\angle APM = 2\angle ADM = 2\theta$$
$$\tan\angle APM = \tan 2\theta = \frac{AM}{PM} \Rightarrow PM = \frac{AM}{\tan 2\theta}$$
$$\tan\angle ADM = \tan\theta = \frac{AM}{DM} \Rightarrow DM = \frac{AM}{\tan\theta}$$

$$\frac{PM}{DM} = \frac{\frac{AM}{\tan 2\theta}}{\frac{AM}{\tan\theta}} = \frac{\tan\theta}{\tan 2\theta} = \frac{\tan\theta}{\frac{2\tan\theta}{1-\tan^2\theta}} = \frac{1}{2}\left(1-\tan^2\theta\right)$$

From (c), $\cos\angle ADB = \frac{7}{65} \Rightarrow \cos 2\theta = \frac{7}{65}$

$$2\cos^2\theta - 1 = \frac{7}{65} \Rightarrow \cos^2\theta = \frac{36}{65} \Rightarrow \sec^2\theta = \frac{65}{36}$$
$$1+\tan^2\theta = \frac{65}{36} \Rightarrow \tan^2\theta = \frac{29}{36}$$

$$\frac{PM}{DM} = \frac{1}{2}\left(1-\frac{29}{36}\right) = \frac{7}{72} \Rightarrow DP : PM = 65 : 7$$

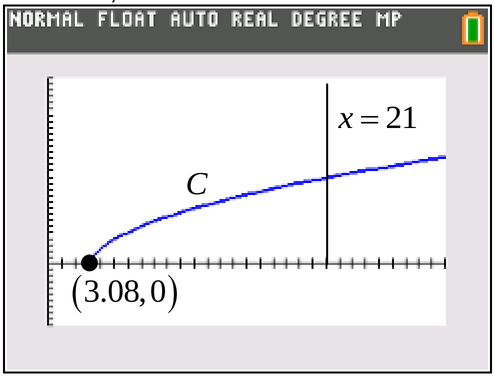
$$\overrightarrow{OM} = \frac{(1)\overrightarrow{OA}+(1)\overrightarrow{OB}}{1+1} = \frac{1}{2}\left[\begin{pmatrix}-1\\2\\5\end{pmatrix}+\begin{pmatrix}1\\-2\\8\end{pmatrix}\right] = \begin{pmatrix}0\\0\\6.5\end{pmatrix}$$

$$\overrightarrow{OP} = \frac{65\overrightarrow{OM}+7\overrightarrow{OD}}{65+7} = \frac{1}{72}\left[65\begin{pmatrix}0\\0\\6.5\end{pmatrix}+7\begin{pmatrix}1\\2\\8.5\end{pmatrix}\right] = \begin{pmatrix}7/72\\7/36\\241/36\end{pmatrix}$$

Question 4

[Ans: (a) $\frac{12152}{75}$ units² (b) $A(5,4)$ (c) $\left(-\frac{16}{5},-\frac{25}{4}\right)$]

(a) From GC,



When $x=21$,

$$2t^2+3=21$$
$$t^2=9$$
$$t=3 \ (\because t\geq \frac{1}{5})$$

$$\frac{dx}{dt}=4t$$

Area

$$=\int_{3.08}^{21} y\,dx$$
$$=\int_{\frac{1}{5}}^{3}(5t-1)(4t)\,dt=\int_{\frac{1}{5}}^{3}20t^2-4t\,dt$$
$$=\left[\frac{20}{3}t^3-2t^2\right]_{\frac{1}{5}}^{3}=\left[\frac{20}{3}(3)^3-2(3)^2\right]-\left[\frac{20}{3}\left(\frac{1}{5}\right)^3-2\left(\frac{1}{5}\right)^2\right]=\frac{12152}{75}$$

(b) Curve C:

$$y=5t-1 \ \Rightarrow t=\frac{y+1}{5} \ ; \qquad x=2\left(\frac{y+1}{5}\right)^2+3 \text{ --- (1)}$$

Curve D:

$$y=\frac{4}{u} \ \Rightarrow u=\frac{4}{y} \ ; \qquad x=5\left(\frac{4}{y}\right) \ \Rightarrow x=\frac{20}{y} \text{ --- (2)}$$

Sub. (2) into (1):

$$\frac{20}{y}=2\left(\frac{y+1}{5}\right)^2+3$$
$$\frac{20}{y}=2\left(\frac{y^2+2y+1}{25}\right)+3$$
$$500=2y^3+4y^2+2y+75y$$
$$2y^3+4y^2+77y-500=0$$

From GC, $y=4$

$$x=\frac{20}{5}=4$$

$\therefore A(5,4)$

Since there is only one solution to (1) and (2), $\therefore$ there are no other points of intersections. (shown)

[continue on next page]

(c) Gradient of tangent to curve C, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{5}{4t}$

At point $A(5,4)$,

$y = 4$

$t = \frac{4+1}{5} = 1$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{4}$

Equation of tangent:

$y - 4 = \frac{5}{4}(x-5)$ --- (3)

For tangent at A to meet curve D,

sub. (2) into (3) $\quad y - 4 = \frac{5}{4}\left(\frac{20}{y} - 5\right)$

$4y^2 + 9y - 100 = 0$

$y = 4$ (NA) or $y = -\frac{25}{4}$

$\therefore x = \frac{20}{-25/4} = -\frac{16}{5}$

$\therefore$ the tangent meets curve D again at $\left(-\frac{16}{5}, -\frac{25}{4}\right)$.

Question 5

[Ans: (a)(i) A and B, A and D (ii) show; B and C (b)(i) A and B, A and D (ii) no]

(a) $P(A)=\frac{18}{36}=\frac{1}{2}$; $P(B)=\frac{18}{36}=\frac{1}{2}$

$P(C)=P(3,6,9,12,15,18,21,24,27,30,33,36)=\frac{12}{36}=\frac{1}{3}$

$P(D)=P(6,12,18,24,30,36)=\frac{6}{36}=\frac{1}{6}$

(i) A and B are mutually exclusive.
A and D are mutually exclusive.

(ii) $P(A\cap C)=\frac{6}{36}=\frac{1}{6}$

$P(A)P(C)=\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)=\frac{1}{6}=P(A\cap C)$

$\therefore A$ and C are independent.

$P(B\cap C)=\frac{6}{36}=\frac{1}{6}$

$P(B)P(C)=\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)=\frac{1}{6}=P(B\cap C)$

$\therefore B$ and C are independent.

(b) $P(A)=\frac{18}{35}$; $P(C)=P(3,6,9,12,15,18,21,24,27,30,33)=\frac{11}{35}$

(i) A and B are mutually exclusive.
A and D are mutually exclusive.

(ii) $P(A\cap C)=\frac{6}{35}$

$P(A)P(C)=\left(\frac{18}{35}\right)\left(\frac{11}{35}\right)=\frac{198}{1225}\neq P(A\cap C)$

$\therefore A$ and C are now not independent.

Question 6

[Ans: (a) show (b) $2r+5=b$; 0.0516]

(a) Probability that Mei obtain 3 red counters $= \dfrac{{}^{r}C_3\,{}^{b}C_9}{{}^{r+b}C_{12}}$

Probability that Mei obtain 4 red counters $= \dfrac{{}^{r}C_4\,{}^{b}C_8}{{}^{r+b}C_{12}}$

$$\frac{{}^{r}C_3\,{}^{b}C_9}{{}^{r+b}C_{12}} = \frac{{}^{r}C_4\,{}^{b}C_8}{{}^{r+b}C_{12}}$$

$$\frac{r!}{3!(r-3)!}\frac{b!}{9!(b-9)!} = \frac{r!}{4!(r-4)!}\frac{b!}{8!(b-8)!}$$

$$4!(r-4)!8!(b-8)! = 3!(r-3)!9!(b-9)!$$

$$4(3!)(r-4)!8!(b-8)(b-9)! = 3!(r-3)(r-4)!9(8!)(b-9)!$$

$$4(b-8) = 9(r-3)$$

$9r+5=4b$ (shown) --- (1)

(b) $\dfrac{{}^{r}C_3\,{}^{b}C_9}{{}^{r+b}C_{12}} = \dfrac{5}{3}\left(\dfrac{{}^{r}C_2\,{}^{b}C_{10}}{{}^{r+b}C_{12}}\right)$

$$\frac{r!}{3!(r-3)!}\frac{b!}{9!(b-9)!} = \frac{5}{3}\left[\frac{r!}{2!(r-2)!}\frac{b!}{10!(b-10)!}\right]$$

$$3\left[2!(r-2)!10!(b-10)!\right] = 5\left[3!(r-3)!9!(b-9)!\right]$$

$$3\left[2!(r-2)(r-3)!10(9!)(b-10)!\right] = 5\left[3(2!)(r-3)!9!(b-9)(b-10)!\right]$$

$$10(r-2) = 5(b-9)$$

$2r+5=b$ --- (2)

Solving (1) & (2) from GC, $r=15$, $b=35$

Probability that Mei obtain 1 red counters

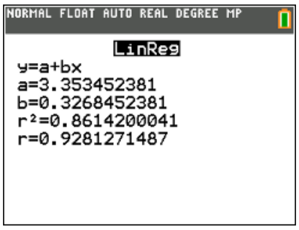
$$= \frac{{}^{15}C_1\,{}^{35}C_{11}}{{}^{15+35}C_{12}} = 0.0516$$

Question 7

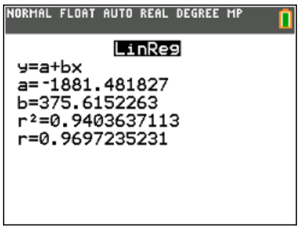
[Ans: (a)(i) $r = 0.9281$ (ii) $r = 0.9697$ (b) $e^y = 376x - 1880$ (c) $y = 8.87$; unreliable]

(a) From GC,

(i) $r = 0.9281$



(ii) $r = 0.9697$



(b) $e^y = cx + d$ gives a better fit as the magnitude of its product moment correlation coefficient is close to 1.

From GC, $e^y = 376x - 1880$

(c) In 2024, $x = 24$,

$$e^y = 375.62(24) - 1881.5$$

$$y = 8.87$$

As $x = 24$ is not within the data range of $4 \le x \le 18$, the estimate is unreliable.

Question 8

[Ans: (a)(i) $H_0 : \mu = 1$, $H_1 : \mu \neq 1$ (b)(i) $H_0 : \mu = 2$, $H_1 : \mu < 2$, $\overline{X} \leq 1.98$
(ii) sufficient evidence]

(a) (i) Let μ be the mean weight of a bag of granulated sugar.

$H_0 : \mu = 1$; $H_1 : \mu \neq 1$

(ii) As the distribution of the weight of a bag of granulated sugar is not known to be following a Normal Distribution and the sample size of 10 is too small for Central Limit Theorem, it is not suitable for z-test .

Furthermore, the selection process described is not random enough so the sample may not be a fair representation of the population.

(b) (i) $H_0 : \mu = 2$

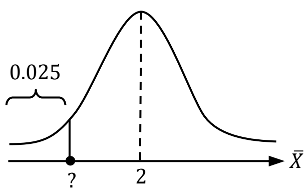
$H_1 : \mu < 2$

$$\sigma^2 \approx s^2 = \frac{1}{n-1}\left[\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right] = \frac{1}{40-1}\left[155.6746 - \frac{(78.88)^2}{40}\right] = 0.00316$$

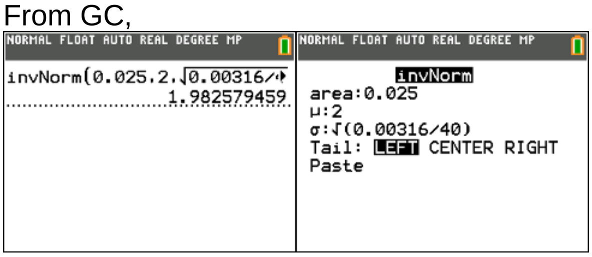
Test Statistics,

Since $n = 40$ (large)

$\overline{X} \sim N\left(2, \frac{0.00316}{40}\right)$ by CLT (approx.)



From GC,



Critical region: $\overline{X} \leq 1.98$

(ii) $n = 40$

$$\bar{x} = \frac{\sum x}{n} = \frac{78.88}{40} = 0.197$$

Since $\bar{x} < 1.98$, it is within the critical region $\therefore$ there is sufficient evidence that average mass of sugar in the bags is less than 2 kg at 2.5% level of significance.

Question 9

[Ans: (a) 0.440 (b) 0.542 (c) 55.2 (d) 0.237 (e) 0.492 (f) explain]

(a) Let X be the length of Long planks.

$X \sim N\left(1.82, 0.2^2\right)$

$P\left(X < 1.79\right) = 0.440$

(b) $E\left(X_1 + \ldots + X_8\right) = 8E\left(X\right) = 8\left(1.82\right) = 14.56$

$Var\left(X_1 + \ldots + X_8\right) = 8Var\left(X\right) = 8\left(0.2^2\right) = 0.32$

$X_1 + \ldots + X_8 \sim N\left(14.56, 0.32\right)$

$P\left(X_1 + \ldots + X_8 > 14.5\right) = 0.542$

(c) Let Y be the length of Regular planks.

$Y \sim N\left(1.22, 0.3^2\right)$

$P\left(Y > 1.25\right) = 0.46017$

Let W be the number of Regular planks that are longer than 1.25 m out of 120 Regular planks.

$W \sim B\left(120, 0.46017\right)$

$E\left(W\right) = 120 \times 0.46017 = 55.2$

(d) $E\left(\left(X_1 + \ldots + X_{10}\right) - \left(Y_1 + \ldots + Y_{16}\right)\right)$

$= 10E\left(X\right) - 16E\left(Y\right) = 10\left(1.82\right) - 16\left(1.22\right) = -1.32$

$Var\left(\left(X_1 + \ldots + X_{10}\right) - \left(Y_1 + \ldots + Y_{16}\right)\right)$

$= 10Var\left(X\right) + 16Var\left(Y\right) = 10\left(0.2^2\right) + 16\left(0.3^2\right) = 1.84$

$\left(X_1 + \ldots + X_{10}\right) - \left(Y_1 + \ldots + Y_{16}\right) \sim N\left(-1.32, 1.84\right)$

$P\left(\left|\left(X_1 + \ldots + X_{10}\right) - \left(Y_1 + \ldots + Y_{16}\right)\right| < 0.65\right)$

$= P\left(-0.65 < \left(X_1 + \ldots + X_{10}\right) - \left(Y_1 + \ldots + Y_{16}\right) < 0.65\right)$

$= 0.237$

(e) Let S_L be the length of a Short planks derived from the Long plank.

$E\left(S_L\right) = E\left(\frac{1}{3}X\right) = \frac{1}{3}E\left(X\right) = \frac{1}{3}\left(1.82\right) = 0.6066667$

$Var\left(S_L\right) = Var\left(\frac{1}{3}X\right) = \left(\frac{1}{3}\right)^2 Var\left(X\right) = \left(\frac{1}{3}\right)^2 \left(0.2^2\right) = 0.0044444$

$S_L \sim N\left(0.6066667, 0.0044444\right)$

[continue on next page]

Let S_R be the length of a Short planks derived from the Regular plank.

$$E(S_R)=E\left(\frac{1}{2}Y\right)=\frac{1}{2}E(Y)=\frac{1}{2}(1.22)=0.61$$

$$Var(S_R)=Var\left(\frac{1}{2}Y\right)=\left(\frac{1}{2}\right)^2 Var(Y)=\left(\frac{1}{2}\right)^2\left(0.3^2\right)=0.0225$$

$$S_R \sim N(0.61, 0.0225)$$

$$E(S_L-S_R)=E(S_L)-E(S_R)=0.6066667-0.61=-0.0033333$$

$$Var(S_L-S_R)=Var(S_L)+Var(S_R)=0.0044444+0.0225=0.026944$$

$$S_L-S_R \sim N(-0.0033333, 0.026944)$$

$$P(S_L>S_R)$$

$$=P(S_L-S_R>0)=0.492$$

(f) As a Long plank will be cut more than a Regular plank to get Short planks, there will be more of of the Long plank that will go to waste, causing possibly a smaller chance that the length of a Short plank made from a Long plank to be greater than that made from a Regular plank. Therefore the probability in part (e) may turn out smaller in this case.

Question 10

[Ans: (a)(i) State (ii) show (iii) 0.677 (iv) 0.805 (v) 0.583 (b) show]

(a) (i) Assumption 1: The probability of any ornament being faulty is the same.
Assumption 2: For one ornament to be faulty, it is independent of any other ornament.

(ii) Let X be the number of faulty ornaments in a day.

$X \sim B(50, 0.04)$

$|E(X) - Var(X)| = |50(0.04) - 50(0.04)(1-0.04)| = 0.08$ (shown)

(iii) $P(X \le 2) = 0.677$

(iv) Let Y be the number of days with no more than 2 faulty ornaments produced in a day out of 5 days.

$Y \sim B(5, 0.67671)$

$P(Y \ge 3) = 1 - P(Y \le 2) = 0.805$

(v) Let W be the number of faulty ornaments in 5 days.

$W \sim B(5 \times 50, 0.04) \Rightarrow W \sim B(250, 0.04)$

$P(W \le 10) = 0.583$

(b) Probability that Mr Lu accepts a box

$= p^6 + \binom{6}{5} p^5 (1-p)$

$= p^6 + 6p^5(1-p) = 6p^5 - 5p^6$

Probability that Mrs Ming accepts a box

$= p^3 + \binom{3}{2} p^2 (1-p) \times p^3$

$= p^3 + 3p^5(1-p) = p^3 + 3p^5 - 3p^6$

Probability that Mrs Ming accepts a box $-$ Probability that Mr Lu accepts a box

$= (p^3 + 3p^5 - 3p^6) - (6p^5 - 5p^6)$

$= p^3 - 3p^5 + 2p^6$

$= p^3(1 - 3p^2 + 2p^3)$

$= p^3(p-1)^2(2p+1) > 0$ $(\because p^3 > 0,\ (p-1)^2 > 0,\ 2p+1 > 0)$

$\therefore$ Mrs Ming accepts a greater proportion of boxes than Mr Lu does. (shown)