O-LEVEL ADDITIONAL MATH 2023 – PAPER 1

Question 1

[Ans: $c=\frac{3}{4}$]

$$x^2+3x+1=2x+c$$
$$x^2+x+1-c=0$$

Discriminant
$$1^2-4(1-c)=0$$
$$1-4+4c=0$$
$$4c=3$$
$$c=\frac{3}{4}$$

Question 2

[Ans: $\frac{3}{x-1}-\frac{5}{x+2}+\frac{4}{(x+2)^2}$]

Let $\frac{18+11x-2x^2}{(x-1)(x+2)^2}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$
$$18+11x-2x^2=A(x+2)^2+B(x-1)(x+2)+C(x-1)$$

When $x=1$,
$$18+11-2=A(1+2)^2$$
$$27=9A$$
$$A=3$$

When $x=-2$,
$$18+11(-2)-2(-2)^2=C(-2-1)$$
$$-12=-3C$$
$$C=4$$

When $x=0$,
$$18=A(2)^2+B(-1)(2)+C(-1)$$
$$18=3(2)^2+-2B+4(-1)$$
$$18=8-2B$$
$$2B=-10$$
$$B=-5$$

$$\frac{18+11x-2x^2}{(x-1)(x+2)^2}=\frac{3}{x-1}-\frac{5}{x+2}+\frac{4}{(x+2)^2}$$

Question 3

[Ans: Prove]

$$LHS = \frac{\cos^2\theta}{(\operatorname{cosec}\theta - 1)(\operatorname{cosec}\theta + 1)} + \frac{\sin^2\theta}{(\sec\theta - 1)(\sec\theta + 1)}$$
$$= \frac{\cos^2\theta}{\operatorname{cosec}^2\theta - 1} + \frac{\sin^2\theta}{\sec^2\theta - 1}$$
$$= \frac{\cos^2\theta}{1 + \cot^2\theta - 1} + \frac{\sin^2\theta}{1 + \tan^2\theta - 1}$$
$$= \frac{\cos^2\theta}{\cot^2\theta} + \frac{\sin^2\theta}{\tan^2\theta}$$
$$= \cos^2\theta\tan^2\theta + \sin^2\theta\cot^2\theta$$
$$= \cos^2\theta\left(\frac{\sin^2\theta}{\cos^2\theta}\right) + \sin^2\theta\left(\frac{\cos^2\theta}{\sin^2\theta}\right)$$
$$= \sin^2\theta + \cos^2\theta$$
$$= 1(RHS)$$

Question 4

[Ans: (a) $e^{-2x}(1-2x)$ (b) $-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + E$]

(a) $\frac{d}{dx}(xe^{-2x})$

$$= (1)(e^{-2x}) + (e^{-2x})(-2)(x)$$
$$= e^{-2x} - 2xe^{-2x}$$
$$= e^{-2x}(1-2x)$$

(b) $\int\left(e^{-2x} - 2xe^{-2x}\right)dx = xe^{-2x} + C$

$$\int e^{-2x}\,dx - \int 2xe^{-2x}\,dx = xe^{-2x} + C$$
$$\frac{e^{-2x}}{-2} - 2\int xe^{-2x}\,dx = xe^{-2x} + D$$
$$-2\int xe^{-2x}\,dx = xe^{-2x} + \frac{e^{-2x}}{2} + D$$
$$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \frac{e^{-2x}}{4} + E$$
$$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + E$$

Queston 5

[Ans: $\theta = -0.615$ or $\theta = 0.615$]

$$\frac{\cos\theta + 4\sin\theta}{2\cos\theta + \sin\theta} = \cot\theta$$

$$\frac{\cos\theta + 4\sin\theta}{2\cos\theta + \sin\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\sin\theta\cos\theta + 4\sin^2\theta = 2\cos^2\theta + \sin\theta\cos\theta$$

$$4\sin^2\theta = 2\cos^2\theta$$

$$2\sin^2\theta = \cos^2\theta$$

$$2\sin^2\theta = 1 - \sin^2\theta$$

$$3\sin^2\theta = 1$$

$$\sin^2\theta = \frac{1}{3}$$

$$\sin\theta = \pm\frac{1}{\sqrt{3}}$$

Ref $\angle = \sin^{-1}\frac{1}{\sqrt{3}} = 0.61548$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta = -0.615$ or $\theta = 0.615$

Question 6

[Ans: (a) $\dfrac{ax(x-2a)}{(x-a)^2}$ (b) $a=4$]

(a) $\mathrm{f}'(x) = \dfrac{(2ax)(x-a)-(1)(ax^2)}{(x-a)^2}$

$= \dfrac{2ax^2-2a^2x-ax^2}{(x-a)^2}$

$= \dfrac{ax^2-2a^2x}{(x-a)^2}$

$= \dfrac{ax(x-2a)}{(x-a)^2}$

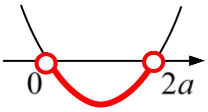
(b) $\mathrm{g}'(x) = (x-a)^2 \times \dfrac{ax(x-2a)}{(x-a)^2}$

$= ax(x-2a)$

$\mathrm{g}(x)$ decreases when $\mathrm{g}'(x) < 0$

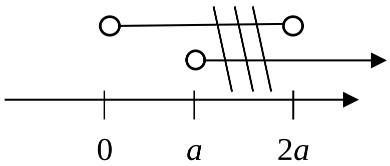
$ax(x-2a) < 0$

Given that a is positive,



$0 < x < 2a$

Given that g is defined for $x > a$,



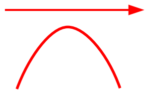
$\therefore a < x < 2a$

Given that $\mathrm{g}(x)$ decreases for $a < x < 8$ $\Rightarrow a = 4$

Question 7

[Ans: $k < -1$]

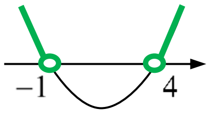
$kx^2 + 4x + k - 3 < 0$



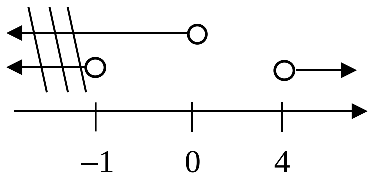
Coefficient of $x^2 < 0$ **and** discriminant < 0

$k < 0$ **and** $4^2 - 4k(k-3) < 0$

$$4^2 - 4k(k-3) < 0$$
$$16 - 4k^2 + 12k < 0$$
$$k^2 - 3k - 4 > 0$$
$$(k-4)(k+1) > 0$$



$k < -1$ or $k > 4$



$\therefore k < -1$

Question 8

[Ans: $(-3, 6)$ and $(-9, -6)$]

$y = 12 + 2x$.....(1)

$x^2 - xy + y^2 = 63$.....(2)

$$x^2 - x(12+2x) + (12+2x)^2 = 63$$
$$x^2 - 12x - 2x^2 + 144 + 48x + 4x^2 = 63$$
$$3x^2 + 36x + 81 = 0$$
$$x^2 + 12x + 27 = 0$$
$$(x+3)(x+9) = 0$$
$$x = -3 \text{ or } x = -9$$
$$y = 12 + 2(-3) = 6$$
$$y = 12 + 2(-9) = -6$$

$(-3, 6)$ and $(-9, -6)$

Question 9

[Ans: (a) $x=\frac{1}{3}$ or $x=-\frac{1}{2}$ (b) Show]

(a) $\frac{\mathrm{d}y}{\mathrm{d}x}=1-x-6x^2$

At stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$.

$1-x-6x^2=0$

$6x^2+x-1=0$

$(3x-1)(2x+1)=0$

$x=\frac{1}{3}$ or $x=-\frac{1}{2}$

(b) Gradient, $m=1-x-6x^2$

Gradient is maximum when $\frac{\mathrm{d}m}{\mathrm{d}x}=0$.

$\frac{\mathrm{d}m}{\mathrm{d}x}=-1-12x$

$0=-1-12x$

$12x=-1$

$x=-\frac{1}{12}$

x-coordinate of P is $-\frac{1}{12}$

x-coordinate for midpoint of $AB=\frac{\left(-\frac{1}{3}\right)+\left(-\frac{1}{2}\right)}{2}=-\frac{1}{12}$

P and the midpoint of AB have the same x-coordinate

Question 10

[Ans: (a) $C(-5,12)$; Radius $=13$ (b) $(0,0)$ and $(0,24)$ (c) $X(-20,48)$ or $X(10,-24)$]

(a) $x^2+10x+y^2-24y=0$

$(x+5)^2-25+(y-12)^2-144=0$

$(x+5)^2+(y-12)^2=169$

$C(-5,12)$; Radius $=\sqrt{169}=13$

(b) Intersects y-axis, $x=0$

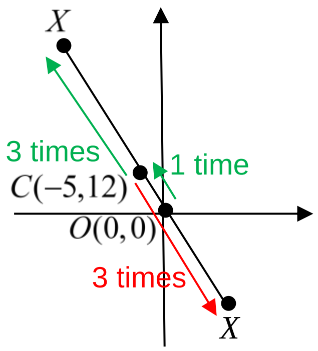
$y^2-24y=0$

$y(y-24)=0$

$y=0$ or $y=24$

$\therefore (0,0)$ and $(0,24)$

(c)



$CX=3OC$

$O\rightarrow C \quad \triangle x=-5, \triangle y=+12$

$C\rightarrow X \quad X\left(-5+3(-5), 12+3(12)\right) \quad \therefore X(-20,48)$

$C\rightarrow X \quad X\left(-5-3(-5), 12-3(12)\right) \quad \therefore X(10,-24)$

Question 11

[Ans: (a) Show (b) $r=\frac{1}{4}\ln(2t+1)+1$ (c) 0.667 cm²/s]

(a) Given $\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{k}{2t+1}$ and $r=1,\frac{\mathrm{d}r}{\mathrm{d}t}=0.5$

Initially, $t=0$.

$$0.5=\frac{k}{2(0)+1}$$

$$k=0.5$$

(b) $r=\int\frac{0.5}{2t+1}\,\mathrm{d}t$

$$=\frac{0.5\ln(2t+1)}{2}+c$$

$$=\frac{1}{4}\ln(2t+1)+c$$

$$t=0,r=1$$

$$1=\frac{1}{4}\ln(2(0)+1)+c$$

$$1=\frac{1}{4}\ln 1+c$$

$$c=1$$

$$r=\frac{1}{4}\ln(2t+1)+1$$

(c)

$$t=3,\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{0.5}{2(3)+1}=\frac{1}{14}$$

$$t=3,r=\frac{1}{4}\ln\left[2(3)+1\right]+1=\frac{1}{4}\ln 7+1$$

$$A=\pi r^2\Rightarrow\frac{\mathrm{d}A}{\mathrm{d}r}=2\pi r=2\pi\left(\frac{1}{4}\ln 7+1\right)$$

$$\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}t}$$

$$=2\pi\left(\frac{1}{4}\ln 7+1\right)\times\frac{1}{14}$$

$$=0.667$$

Question 12

[Ans: (a) 1.75 m (b) $3-5\left(t-\frac{1}{2}\right)^2$ (c) 3 m ; 0.5 s (d) Explain (e) 0.894 s]

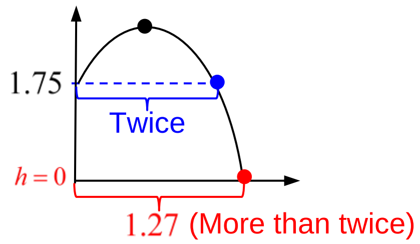
(a) $t=0,\ h=1.75$

(b) $h=-5t^2+5t+1.75$
$=-5(t^2-t)+1.75$
$=-5\left[\left(t-\frac{1}{2}\right)^2-\frac{1}{4}\right]+1.75$
$=-5\left(t-\frac{1}{2}\right)^2+\frac{5}{4}+1.75$
$=3-5\left(t-\frac{1}{2}\right)^2$

(c) Maximum height $=3\text{m}$, time at which this occurs $=0.5\text{ s}$.

(d) At $t=0$, the ball is not thrown from the ground.
The ball was thrown at the height of 1.75 m.
Hits the ground, $h=0$.

$0=3-5\left(t-\frac{1}{2}\right)^2 \Rightarrow t=1.27$



(e) At $h=2$,

$3-5\left(t-\frac{1}{2}\right)^2=2$

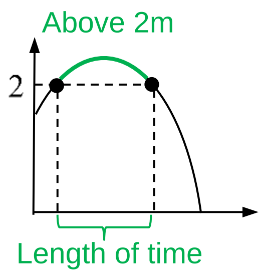
$5\left(t-\frac{1}{2}\right)^2=1$

$\left(t-\frac{1}{2}\right)^2=\frac{1}{5}$

$t-\frac{1}{2}=\pm\frac{1}{\sqrt{5}}$

$t=\frac{1}{2}\pm\frac{1}{\sqrt{5}}$

Length of time
$=\left(\frac{1}{2}+\frac{1}{\sqrt{5}}\right)-\left(\frac{1}{2}-\frac{1}{\sqrt{5}}\right)$
$=\frac{2}{\sqrt{5}}$
$=0.894$



Question 13

[Ans: (a) $-\frac{10}{h+7}$ (b) $h=9$ or $h=-2$]

(a) Gradient of AB,

$$m_{AB}=\frac{h-(-7)}{2-(-8)}=\frac{h+7}{10}$$

Perpendicular Gradient of perpendicular bisector of AB,

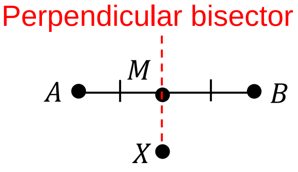
$$m_{\perp}=-1\div\frac{h+7}{10}=-\frac{10}{h+7}$$

(b) Perpendicular bisector passes through $X\left(h,-\frac{13}{2}\right)$

Midpoint of AB,

$$M\left(\frac{2+(-8)}{2},\frac{h+(-7)}{2}\right)$$

$$\therefore M\left(-3,\frac{h-7}{2}\right)$$



Equation of the perpendicular bisector of AB,

$$y-\frac{h-7}{2}=-\frac{10}{h+7}(x+3)$$

$X\left(h,-\frac{13}{2}\right)$ lies on the line,

$$-\frac{13}{2}-\frac{h-7}{2}=-\frac{10}{h+7}(h+3)$$

$$13+(h-7)=\frac{20}{h+7}(h+3)$$

$$h+6=\frac{20h+60}{h+7}$$

$$h^2+13h+42=20h+60$$

$$h^2-7h-18=0$$

$$(h-9)(h+2)=0$$

$$h=9 \text{ or } h=-2$$