A-LEVEL H2 MATH 2022 – PAPER 1

Question 1

[Ans: $z=2+i$, $w=-i$]

$iz+2w=-1$ (1)

$i(iz+2w)=-i$

$-z+2iw=-i$ (2)

$(2-i)z+iw=6$

$2[(2-i)z+iw]=12$

$(4-2i)z+2iw=12$ (3)

(3) − (2)

$$(4-2i)z+z=12+i$$

$$(5-2i)z=12+i$$

$$z=\frac{12+i}{5-2i}\left(\frac{5+2i}{5+2i}\right)$$

$$=\frac{60+24i+5i-2}{5^2+2^2}$$

$$=\frac{58+29i}{29}=2+i$$

Sub. $z=2+i$ into (1)

$$i(2+i)+2w=-1$$

$$2i-1+2w=-1$$

$$2w=-2i$$

$$w=-i$$

Question 2

[Ans: (a) $f'(x)=\dfrac{1}{1+\left(\sqrt{2}+x\right)^2}$, $f''(x)=\dfrac{-2\left(\sqrt{2}+x\right)}{\left[1+\left(\sqrt{2}+x\right)^2\right]^2}$ (b) $0.955+0.333x-0.157x^2+\dots$]

(a) Let $y=f(x)$

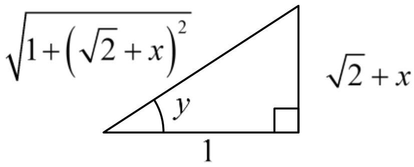
$y=\tan^{-1}\left(\sqrt{2}+x\right)$

$\tan y=\sqrt{2}+x$ (1)

Differentiate (1) w.r.t. x,

$\sec^2 y\dfrac{dy}{dx}=1$

$\dfrac{dy}{dx}=\cos^2 y$ (2)



$\dfrac{dy}{dx}=\dfrac{1}{1+\left(\sqrt{2}+x\right)^2}=f'(x)$

Differentiate (2) w.r.t. x,

$$\frac{d^2y}{dx^2}=2\cos y(-\sin y)\frac{dy}{dx}$$

$$=-2\sin y\cos y\frac{dy}{dx}$$

$$=-2\left[\frac{\sqrt{2}+x}{\sqrt{1+\left(\sqrt{2}+x\right)^2}}\right]\left[\frac{1}{\sqrt{1+\left(\sqrt{2}+x\right)^2}}\right]\left[\frac{1}{1+\left(\sqrt{2}+x\right)^2}\right]$$

$$=\frac{-2\left(\sqrt{2}+x\right)}{\left[1+\left(\sqrt{2}+x\right)^2\right]^2}=f''(x)$$

(b) $f(0)=\tan^{-1}\left(\sqrt{2}\right)$, $f'(0)=\dfrac{1}{1+\left(\sqrt{2}+0\right)^2}=\dfrac{1}{3}$, $f''(0)=\dfrac{-2\left(\sqrt{2}+0\right)}{\left[1+\left(\sqrt{2}+0\right)^2\right]^2}=-\dfrac{2\sqrt{2}}{9}$

$$f(x)=\tan^{-1}\sqrt{2}+\frac{1}{3}x+\frac{-2\sqrt{2}/9}{2!}x^2+\dots$$

$$=\tan^{-1}\sqrt{2}+\frac{1}{3}x-\frac{\sqrt{2}}{9}x^2+\dots=0.955+0.333x-0.157x^2+\dots$$

Question 3

[Ans: (a) $-\frac{1}{3}$ (b) $x^2 - y^2 = 2$, $x \geq \sqrt{2}$]

(a) $\frac{dx}{dt} = \frac{1}{2}\left(3e^{3t} - 6e^{-3t}\right) = \frac{3}{2}\left(e^{3t} - 2e^{-3t}\right)$

$\frac{dy}{dt} = \frac{1}{2}\left(3e^{3t} + 6e^{-3t}\right) = \frac{3}{2}\left(e^{3t} + 2e^{-3t}\right)$

$$\frac{dy}{dx} = \frac{\frac{3}{2}\left(e^{3t} + 2e^{-3t}\right)}{\frac{3}{2}\left(e^{3t} - 2e^{-3t}\right)} = \frac{e^{3t} + 2e^{-3t}}{e^{3t} - 2e^{-3t}}$$

Where $t = \frac{1}{3}\ln 2$,

$$\frac{dy}{dx} = \frac{e^{3\left(\frac{1}{3}\ln 2\right)} + 2e^{-3\left(\frac{1}{3}\ln 2\right)}}{e^{3\left(\frac{1}{3}\ln 2\right)} - 2e^{-3\left(\frac{1}{3}\ln 2\right)}} = \frac{e^{\ln 2} + 2e^{\ln\frac{1}{2}}}{e^{\ln 2} - 2e^{\ln\frac{1}{2}}}$$

$$= \frac{2 + 2\left(\frac{1}{2}\right)}{2 - 2\left(\frac{1}{2}\right)} = 3$$

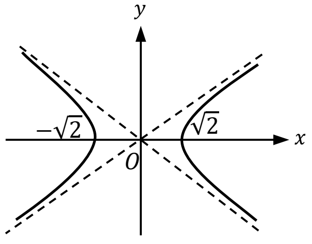
$\therefore$ Gradient of normal where $t = \frac{1}{3}\ln 2$ is $-\frac{1}{3}$.

(b) $x^2 = \frac{1}{4}\left(e^{3t} + 2e^{-3t}\right)^2 = \frac{1}{4}\left(e^{6t} + 4 + 4e^{-6t}\right)$

$y^2 = \frac{1}{4}\left(e^{3t} - 2e^{-3t}\right)^2 = \frac{1}{4}\left(e^{6t} - 4 + 4e^{-6t}\right)$

$x^2 - y^2 = \frac{1}{4}\left(4 + 4\right) = 2$

From graph of $x^2 - y^2 = 2$,



$x \leq -\sqrt{2}$ or $x \geq \sqrt{2}$

And since $x = \frac{1}{2}\left(e^{3t} + 2e^{-3t}\right) > 0$,

$\therefore x \geq \sqrt{2}$

Question 4

[Ans: (a) show (b) show (c) $\frac{1}{6}\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)$]

(a) $\frac{d}{dx}(\cot x)$

$=\frac{d}{dx}(\tan x)^{-1}$

$=-(\tan x)^{-2}\sec^2 x$

$=-\left(\frac{1}{\tan^2 x}\right)\left(\frac{1}{\cos^2 x}\right)=-\left(\frac{\cos^2 x}{\sin^2 x}\right)\left(\frac{1}{\cos^2 x}\right)$

$=-\operatorname{cosec}^2 x$ (shown)

(b) $\sin 2x\tan x$

$=2\sin x\cos x\left(\frac{\sin x}{\cos x}\right)=2\sin^2 x$ (shown)

(c) $\int_{\frac{\pi}{18}}^{\frac{\pi}{9}}\operatorname{cosec}6x\cot 3x\,dx$

$=\int_{\frac{\pi}{18}}^{\frac{\pi}{9}}\frac{1}{[\sin 2(3x)]\tan(3x)}dx$

$=\int_{\frac{\pi}{18}}^{\frac{\pi}{9}}\frac{1}{2\sin^2 3x}dx$

$=-\frac{1}{2}\int_{\frac{\pi}{18}}^{\frac{\pi}{9}}-\operatorname{cosec}^2 3x\,dx$

$=-\frac{1}{2}\left[\frac{\cot 3x}{3}\right]_{\frac{\pi}{18}}^{\frac{\pi}{9}}$

$=-\frac{1}{6}\left[\cot\frac{\pi}{3}-\cot\frac{\pi}{6}\right]=-\frac{1}{6}\left(\frac{1}{\sqrt{3}}-\sqrt{3}\right)=\frac{1}{6}\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)$

Queston 5

[Ans: (a) show (b) $A(-12,8)$, $B\left(-\frac{4}{5},\frac{72}{5}\right)$]

(a) $y=mx$ (1) $(x+8)^2+(y-14)^2=52$ (2)

Sub. (1) into (2)

$(x+8)^2+(mx-14)^2=52$

$x^2+16x+64+m^2x^2-28mx+196=52$

$(m^2+1)x^2(16-28m)x+208=0$

Let Discriminant $=0$

$(16-28m)^2-4(m^2+1)(208)=0$

$256-896m+784m^2-832m^2-832=0$

$-48m^2-896m-576=0 \Rightarrow -16(3m^2+56m+36)=0$

$3m^2+56m+36=0$ (shown)

(b) From GC, $m=-\frac{2}{3}$ or $m=-18$.

When $m=-\frac{2}{3}$,

$(m^2+1)x^2(16-28m)x+208=0$

$\left[\left(-\frac{2}{3}\right)^2+1\right]x^2\left[16-28\left(-\frac{2}{3}\right)\right]x+208=0 \Rightarrow \frac{13}{9}x^2+\frac{104}{3}x+208=0$

$x=-12$, $y=\left(-\frac{2}{3}\right)(-12)=8$

$\therefore A(-12,8)$

When $m=-18$,

$(m^2+1)x^2(16-28m)x+208=0$

$\left[(-18)^2+1\right]x^2\left[16-28(-18)\right]x+208=0 \Rightarrow 325x^2+520x+208=0$

$x=-\frac{4}{5}$, $y=(-18)\left(-\frac{4}{5}\right)=\frac{72}{5}$

$\therefore B\left(-\frac{4}{5},\frac{72}{5}\right)$

Question 6

[Ans: (a) describe (b) $\mathrm{f}^{-1}(x)=\frac{ax+k}{x-a}$ (c) $\mathrm{f}^{2}(x)=x$ (d) $\frac{a+k}{1-a}$]

(a) $\mathrm{f}(x)=\frac{ax+k}{x-a}=\frac{a\left[(x-a)+a\right]+k}{x-a}=a+\frac{a^2+k}{x-a}$

$y=\frac{1}{x}\rightarrow y=\frac{1}{x-a}\rightarrow y=\frac{a^2+k}{x-a}\rightarrow y=a+\frac{a^2+k}{x-a}=\mathrm{f}(x)$

Sequence of transformations:

(1) Transate by a units in the positive x direction

(2) Scale by a factor of $\left(a^2+k\right)$ parallel to the y-axis

(3) Translate by a units in the positive y direction

(b) Let $y=\mathrm{f}(x)$

$y=\frac{ax+k}{x-a}, x\neq a$

$xy-ay=ax+k$

$xy-ax=ay+k$

$x=\frac{ay+k}{y-a}$

$\therefore \mathrm{f}^{-1}(x)=\frac{ax+k}{x-a}$

(c) $\mathrm{f}^{2}(x)$

$=\mathrm{ff}(x)$

$=\mathrm{ff}^{-1}(x)=x$

(d) $\mathrm{f}^{2023}(1)$

$=\mathrm{ff}^{2022}(1)$

$=\mathrm{f}(1)=\frac{a(1)+k}{1-a}=\frac{a+k}{1-a}$

Question 7

[Ans: (a) show; $\left(e^{\frac{1}{3}}, \frac{1}{3e}\right)$ (b) $\left(\frac{2}{9} - \frac{1}{18}\ln 3\right)$ units²]

(a) $y = x^{-3}\ln x$

$$\frac{dy}{dx} = x^{-3}\left(\frac{1}{x}\right) + \left(-3x^{-4}\right)\ln x$$

$$= \frac{1}{x^4} - \frac{3\ln x}{x^4} = \frac{1-3\ln x}{x^4} \text{ (shown)}$$

Let $\frac{dy}{dx} = 0$

$$\frac{1-3\ln x}{x^4} = 0$$

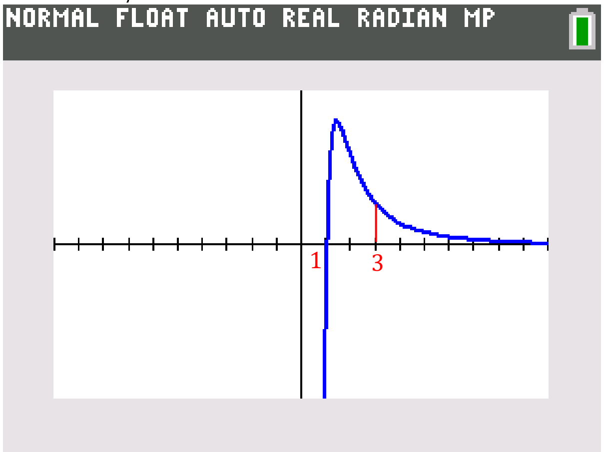
$$1 - 3\ln x = 0$$

$$\ln x = \frac{1}{3} \Rightarrow x = e^{\frac{1}{3}}$$

$$y = \left(e^{\frac{1}{3}}\right)^{-3}\ln\left(e^{\frac{1}{3}}\right) = \frac{1}{3e}$$

$\therefore$ Coordinates of turning point of C is $\left(e^{\frac{1}{3}}, \frac{1}{3e}\right)$.

(b) From GC,



Area

$$= \int_1^3 x^{-3}\ln x\,dx$$

Let $u = \ln x$, $\frac{dv}{dx} = x^{-3}$

$$\frac{du}{dx} = \frac{1}{x},\quad v = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$$

$$= \left[(\ln x)\left(-\frac{1}{2x^2}\right)\right]_1^3 - \int_1^3 \left(\frac{1}{x}\right)\left(-\frac{1}{2x^2}\right)dx$$

$$= \left[\left(-\frac{1}{18}\ln 3\right) - (0)\right] + \frac{1}{2}\int_1^3 x^{-3}dx$$

$$= -\frac{1}{18}\ln 3 + \frac{1}{2}\left[\frac{x^{-2}}{-2}\right]_1^3$$

$$= -\frac{1}{18}\ln 3 - \frac{1}{4}\left(\frac{1}{9} - 1\right)$$

$$= \frac{2}{9} - \frac{1}{18}\ln 3$$

Question 8

[Ans: (a) $\ln\left(x^2+2x+1\right)+\dfrac{3}{x+1}+C$ (b) $2\ln\dfrac{4}{3}$]

(a) $\displaystyle\int \frac{2x-1}{x^2+2x+1}\,dx$

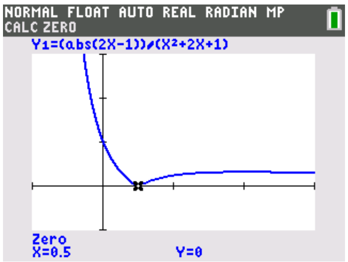
$$=\int \frac{(2x+2)-2-1}{x^2+2x+1}\,dx$$

$$=\int \frac{2x+2}{x^2+2x+1}-\frac{3}{(x+1)^2}\,dx=\int \frac{2x+2}{x^2+2x+1}-3(x+1)^{-2}\,dx$$

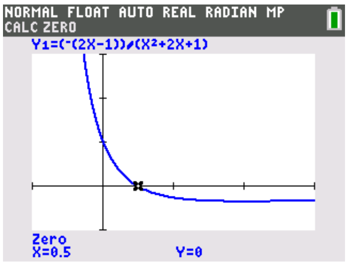
$$=\ln\left(x^2+2x+1\right)-3\left[\frac{(x+1)^{-1}}{-1}\right]+C=\ln\left(x^2+2x+1\right)+\frac{3}{x+1}+C$$

(b) From GC,

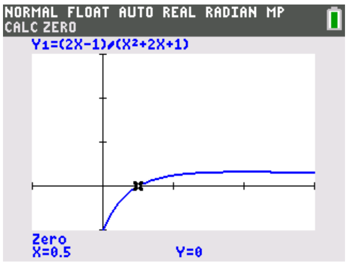
$y=\dfrac{|2x-1|}{x^2+2x+1}$



$y=\dfrac{-(2x-1)}{x^2+2x+1}$



$y=\dfrac{2x-1}{x^2+2x+1}$



$$\int_0^2 \frac{|2x-1|}{x^2+2x+1}\,dx$$

$$=\int_0^{\frac{1}{2}} \frac{-(2x-1)}{x^2+2x+1}\,dx+\int_{\frac{1}{2}}^2 \frac{2x-1}{x^2+2x+1}\,dx$$

$$=-\left[\ln\left(x^2+2x+1\right)+\frac{3}{x+1}\right]_0^{\frac{1}{2}}+\left[\ln\left(x^2+2x+1\right)+\frac{3}{x+1}\right]_{\frac{1}{2}}^2$$

$$=-\left[\left(\ln\frac{9}{4}+2\right)-(3)\right]+\left[(\ln 9+1)-\left(\ln\frac{9}{4}+2\right)\right]$$

$$=-\ln\frac{9}{4}+1+\ln 9-\ln\frac{9}{4}-1$$

$$=\ln\left(\frac{9}{\frac{9}{4}\times\frac{9}{4}}\right)=\ln\frac{16}{9}=2\ln\frac{4}{3}$$

Question 9

[Ans: (a) $d=\frac{5}{2}a$ (b)(i) show, $k=\frac{1}{2}$ (ii) $\frac{43}{128}\sqrt{3}$]

(a) Given

	AP	GP	
(1)	a	a	(1)
(3)	$a+2d$	ar	(2)
(15)	$a+14d$	ar^2	(3)

$$r=\frac{a+14d}{a+2d}=\frac{a+2d}{a}$$
$$a\left(a+14d\right)=\left(a+2d\right)^2$$
$$a^2+14ad=a^2+4ad+4d^2$$
$$4d^2-10ad=0$$
$$2d\left(2d-5a\right)=0$$
$$d=\frac{5}{2}a$$

(b) (i) Sum to infinity

$$=\frac{\sin\theta}{1-\left(-\cos\theta\right)}=\frac{\sin\theta}{1+\cos\theta}$$

$$=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\tan\frac{\theta}{2}$$ (shown), $k=\frac{1}{2}$

(ii) For $\theta=\frac{\pi}{3}$,

first term $=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$, common ratio $=-\cos\frac{\pi}{3}=-\frac{1}{2}$

Sum of first seven terms

$$=\frac{\left(\frac{\sqrt{3}}{2}\right)\left[1-\left(-\frac{1}{2}\right)^7\right]}{1-\left(-\frac{1}{2}\right)}=\frac{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{129}{128}\right)}{\frac{3}{2}}=\frac{43}{128}\sqrt{3}$$

Question 10

[Ans: (a) $a+2b<0$ (b) sketch (c) sketch (d) $x\le 2$ or $x>1$]

(a) $y=ax+b+\dfrac{a+2b}{x-1}$

$\dfrac{dy}{dx}=a-\dfrac{a+2b}{(x-1)^2}$

Let $\dfrac{dy}{dx}=0$,

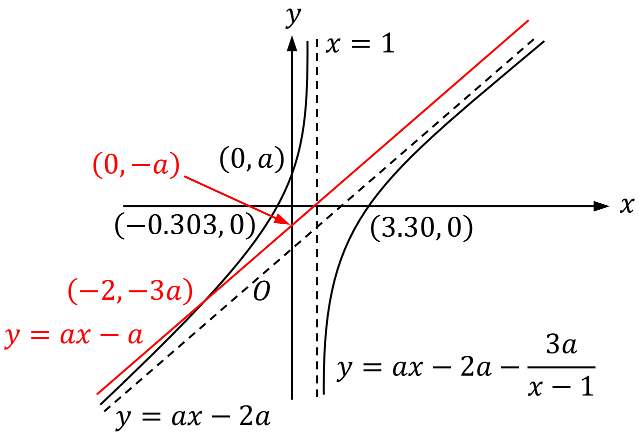
$a-\dfrac{a+2b}{(x-1)^2}=0$

$(x-1)^2=\dfrac{a+2b}{a}$

Let $\dfrac{a+2b}{a}<0$

$a+2b<0$ ($\because a>0$ and $b\ne -\dfrac{1}{2}a$)

(b) $y=ax+b+\dfrac{a+2b}{x-1}=ax+(-2a)+\dfrac{a+2(-2a)}{x-1}=ax-2a-\dfrac{3a}{x-1}$



(c) As shown in graph

(d) $x-2-\dfrac{3}{x-1}\le x-1\Rightarrow (1)x-2(1)-\dfrac{3(1)}{x-1}\le (1)x-(1)$

From graph, $x\le 2$ or $x>1$

Question 11

[Ans: (a) $p=-36$ (b) $5x+y+2z=2560$ (c) $\left(512,44,-22\right)$ (d) $1.3°$]

(a) $\overrightarrow{PQ}$

$=\overrightarrow{OQ}-\overrightarrow{OP}$

$$=\begin{pmatrix}200\\20\\-15\end{pmatrix}-\begin{pmatrix}1136\\92\\p\end{pmatrix}=\begin{pmatrix}-936\\-72\\-15-p\end{pmatrix}$$

$\left|\overrightarrow{PQ}\right|=939$

$\sqrt{\left(-936\right)^2+\left(-72\right)^2+\left(-15-p\right)^2}=939$

$881280+225+30p+p^2=881721$

$p^2+30p-216=0$

$p=-36$ or $p=6$ (NA)

(b) A vector parallel to this plane $=\begin{pmatrix}500\\200\\-70\end{pmatrix}-\begin{pmatrix}400\\600\\-20\end{pmatrix}=-50\begin{pmatrix}-2\\8\\1\end{pmatrix}$

Another vector parallel to this plane $=\begin{pmatrix}600\\-340\\-50\end{pmatrix}-\begin{pmatrix}400\\600\\-20\end{pmatrix}=10\begin{pmatrix}20\\-94\\-3\end{pmatrix}$

Vector perpendicular to this plane $=\begin{pmatrix}-2\\8\\1\end{pmatrix}\times\begin{pmatrix}20\\-94\\-3\end{pmatrix}=\begin{pmatrix}70\\14\\28\end{pmatrix}=14\begin{pmatrix}5\\1\\2\end{pmatrix}$

Equation of this plane: $\mathbf{r}\cdot\begin{pmatrix}5\\1\\2\end{pmatrix}=\begin{pmatrix}400\\600\\-20\end{pmatrix}\cdot\begin{pmatrix}5\\1\\2\end{pmatrix}=2560\Rightarrow 5x+y+2z=2560$

(c) For $p=-36$, $\overrightarrow{PQ}=\begin{pmatrix}-936\\-72\\-15-\left(-36\right)\end{pmatrix}=\begin{pmatrix}-936\\-72\\21\end{pmatrix}$

Equation of pipeline: $\mathbf{r}=\begin{pmatrix}200\\20\\-15\end{pmatrix}+\lambda\begin{pmatrix}-936\\-72\\21\end{pmatrix}=\begin{pmatrix}200-936\lambda\\20-72\lambda\\-15+21\lambda\end{pmatrix}$

$5\left(200-936\lambda\right)+\left(20-72\lambda\right)+2\left(-15+21\lambda\right)=2560$

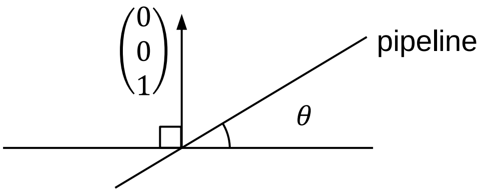
$-4710\lambda=1570\Rightarrow\lambda=-\dfrac{1}{3}$

Position vector where the pipe meets the rock

$$=\begin{pmatrix} 200-936\left(-\frac{1}{3}\right) \\ 20-72\left(-\frac{1}{3}\right) \\ -15+21\left(-\frac{1}{3}\right) \end{pmatrix}=\begin{pmatrix} 512 \\ 44 \\ -22 \end{pmatrix}$$

$\therefore$ the pipe meets the rock at $(512, 44, -22)$.

(d)



$$\left|\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} -936 \\ -72 \\ 21 \end{pmatrix}\right|=\left|\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|\left|\begin{pmatrix} -936 \\ -72 \\ 21 \end{pmatrix}\right|\cos(90°-\theta)$$

$$\cos(90°-\theta)=\frac{21}{939}$$

$$\theta=90°-\cos^{-1}\left(\frac{21}{939}\right)=1.3°$$

Question 12

[Ans: (a) $P=50e^{\frac{1}{10}t\ln 2}$ (b) $P=\dfrac{500}{9e^{-\frac{1}{10}t\ln\frac{9}{4}}+1}$ (c) $P\to 500$; comment]

(a) $\dfrac{dP}{dt}=kP$

$\int \dfrac{1}{P}\,dP=k\int dt$

$\ln|P|=kt+A$

$|P|=e^{A}e^{kt}\Rightarrow P=Be^{kt},\ B=\pm e^{A}$

When $t=0$,

$P=50$

$Be^{0}=50\Rightarrow B=50$

When $t=10$,

$P=100$

$50e^{10k}=100\Rightarrow e^{10k}=2\Rightarrow 10k=\ln 2\Rightarrow k=\dfrac{1}{10}\ln 2$

$P=50e^{\frac{1}{10}t\ln 2}$

(b) $\dfrac{dP}{dt}=\lambda P(500-P)$

$\int \dfrac{1}{P(500-P)}\,dP=\lambda\int dt$

$\int \dfrac{1}{500P}+\dfrac{1}{500(500-P)}\,dP=\lambda\int dt$

$\dfrac{1}{500}\left(\ln|P|-\ln|500-P|\right)=\lambda t+C$

$\ln\left|\dfrac{P}{500-P}\right|=500\lambda t+500C$

$\left|\dfrac{P}{500-P}\right|=e^{500C}e^{500\lambda t}\Rightarrow \dfrac{P}{500-P}=De^{500\lambda t},\ D=\pm e^{500C}$

When $t=0$, $P=50$,

$\dfrac{50}{500-50}=De^{0}\Rightarrow D=\dfrac{1}{9}$

When $t=10$, $P=100$

$\dfrac{100}{500-100}=\dfrac{1}{9}e^{500\lambda(10)}\Rightarrow e^{5000\lambda}=\dfrac{9}{4}$

$5000\lambda=\ln\dfrac{9}{4}\Rightarrow \lambda=\dfrac{1}{5000}\ln\dfrac{9}{4}$

$$\frac{P}{500-P}=\frac{1}{9}e^{500\left(\frac{1}{5000}\ln\frac{9}{4}\right)t}\Rightarrow\frac{P}{500-P}=\frac{1}{9}e^{\frac{1}{10}t\ln\frac{9}{4}}$$

$$9Pe^{-\frac{1}{10}t\ln\frac{9}{4}}=500-P$$

$$9Pe^{-\frac{1}{10}t\ln\frac{9}{4}}+P=500\Rightarrow P=\frac{500}{9e^{-\frac{1}{10}t\ln\frac{9}{4}}+1}$$

(c) When $t\to\infty$,

from (d), $P\to\frac{500}{0+1}=500$

When $t\to\infty$,

from (a), $P\to\infty$

Based on the expected long run population deduced from the two models, the refined model is an improvement as it is more likely that the population of the insect species will start to become stagnant at a certain value instead of increasing indefinitely.