O-LEVEL ADDITIONAL MATH 2022 – PAPER 2

Question 1

[Ans: (a) 1660 birds (b) Year 2023]

(a)

Year 2010, $t = 0$.

Year 2013, $t = 3, N = 2050$.

$$2050 = 2400\,\mathrm{e}^{-3k}$$
$$\mathrm{e}^{-3k} = \frac{2050}{2400} = \frac{41}{48}$$
$$-3k = \ln\frac{41}{48}$$
$$k = -\frac{1}{3}\ln\frac{41}{48}$$
$$= -0.05254$$

Year 2017, $t = 7$.

$$N = 2400\,\mathrm{e}^{(-0.05254)(7)}$$
$$= 1661.4$$

$\therefore$ About 1660 (3 sf) birds.

(b)

$$N < 1200$$
$$2400\,\mathrm{e}^{-0.05254t} < 1200$$
$$\mathrm{e}^{-0.05254t} < 0.5$$
$$-0.05254t < \ln 0.5$$
$$t > \frac{\ln 0.5}{-0.05254}$$
$$t > 13.19$$

Checking:

$N = 2400\,\mathrm{e}^{(-0.05254)(13)} = 1212.2$ (Year 2023, Jan 1)

$N = 2400\,\mathrm{e}^{(-0.05254)(14)} = 1150.2$ (Year 2024, Jan 1)

First labelled “under threat” in year 2023.

Question 2

[Ans: (a) $a = 2, b = -6$ (b) $\mathrm{f}(x) = (x+2)(x+\sqrt{3})(x-\sqrt{3})$]

(a)

$\mathrm{f}(-2) = 0,$

$(-2)^3 + a(-2)^2 - 3(-2) + b = 0$

$4a + b - 2 = 0$

$4a + b = 2 \text{.........}(1)$

$\mathrm{f}(3) = 30,$

$(3)^3 + a(3)^2 - 3(3) + b = 30$

$9a + b + 18 = 30$

$9a + b = 12 \text{.........}(2)$

$(2) - (1):$

$5a = 10$

$a = 2$

Sub into (1):

$4(2) + b = 2$

$b = -6$

$\therefore a = 2, b = -6$

(b)

$\mathrm{f}(x) = x^3 + 2x^2 - 3x - 6$

$= (x+2)(Ax^2 + Bx + C)$

$= (x+2)(x^2 + Bx - 3)$

Comparing coefficients of x^2

$B + 2 = 2 \quad \Rightarrow B = 0$

$\mathrm{f}(x) = (x+2)(x^2 - 3)$

$= (x+2)(x+\sqrt{3})(x-\sqrt{3})$

Question 3

[Ans: (a) Prove (b) $\theta = 15°, 75°, 195°, 255°$]

(a)

$$\begin{aligned} LHS &= \tan 2\theta(2\cos\theta - \sec\theta) \\ &= \frac{\sin 2\theta}{\cos 2\theta}\left(2\cos\theta - \frac{1}{\cos\theta}\right) \\ &= \frac{2\sin\theta\cos\theta}{2\cos^2\theta - 1}\left(\frac{2\cos^2\theta - 1}{\cos\theta}\right) \\ &= 2\sin\theta (RHS) \end{aligned}$$

(b)

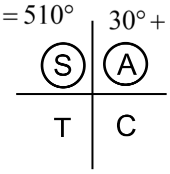
$$\begin{aligned} \tan 2\theta(2\cos\theta - \sec\theta) &= \frac{1}{2}\sec\theta \\ 2\sin\theta &= \frac{1}{2}\sec\theta \\ 2\sin\theta &= \frac{1}{2}\left(\frac{1}{\cos\theta}\right) \\ 2\sin\theta\cos\theta &= \frac{1}{2} \\ \sin 2\theta &= \frac{1}{2} \\ 2\theta &= \sin^{-1}\frac{1}{2}, \quad 0° \le 2\theta \le 720° \\ 2\theta &= 30°, 150°, 390°, 510° \\ \theta &= 15°, 75°, 195°, 255° \end{aligned}$$

$180° - 30° = 150°$

$150° + 360° = 510°$

Ref $\angle = 30°$

$30° + 360° = 390°$



Question 4

[Ans: (a) Show (b) 7.63 (c) Minimum]

(a) Volume

$$V = 3x^2h$$

$$2000 = 3x^2h$$

$$h = \frac{2000}{3x^2}$$

Total surface area

$$A = 2(3x)(x) + 2(x)(h) + 2(3x)(h)$$

$$= 6x^2 + 2hx + 6hx$$

$$= 6x^2 + 8hx$$

$$= 6x^2 + 8\left(\frac{2000}{3x^2}\right)x$$

$$= 6x^2 + \frac{16000}{3x}$$

(Shown)

(b)

$$A = 6x^2 + \frac{16000}{3}x^{-1}$$

$$\frac{dA}{dx} = 12x - \frac{16000}{3}x^{-2}$$

At stationary A, $\frac{dA}{dx} = 0$

$$12x - \frac{16000}{3x^2} = 0$$

$$12x^3 - \frac{16000}{3} = 0$$

$$12x^3 = \frac{16000}{3}$$

$$x^3 = \frac{4000}{9}$$

$$x = \sqrt[3]{\frac{4000}{9}}$$

$x \approx 7.63$ (3 sf)

(c)

$$\frac{d^2A}{dx^2} = 12 + \frac{32000}{3}x^{-3}$$

$$= 12 + \frac{32000}{3}\left(\sqrt[3]{\frac{4000}{9}}\right)^{-3}$$

$$= 36$$

$\because \frac{d^2A}{dx^2} > 0$, minimum value.

Queston 5

[Ans: (a) Show (b) $8+4\sqrt{3}$ (c) $\frac{4}{26}-\frac{1}{26}\sqrt{3}$]

(a)

$$\tan 75° = \tan(45°+30°)$$
$$= \frac{\tan 45° + \tan 30°}{1-\tan 45° \tan 30°}$$
$$= \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}$$
$$= \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$
$$= \frac{3+2\sqrt{3}+1}{3-1}$$
$$= \frac{4+2\sqrt{3}}{2}$$
$$= 2+\sqrt{3}$$

From calculator

$$(\tan 30°)^2 = \frac{1}{3}$$
$$\tan 30° = \frac{1}{\sqrt{3}}$$

(b)

$$\sec^2 75° = 1+\tan^2 75°$$
$$= 1+(2+\sqrt{3})^2$$
$$= 1+4+4\sqrt{3}+3$$
$$= 8+4\sqrt{3}$$

(c)

$$\cos^2 75° = \frac{1}{\sec^2 75°}$$
$$= \frac{1}{8+4\sqrt{3}} \times \frac{8-4\sqrt{3}}{8-4\sqrt{3}}$$
$$= \frac{8-4\sqrt{3}}{64-16(3)}$$
$$= \frac{8-4\sqrt{3}}{16}$$
$$= \frac{1}{2}-\frac{1}{4}\sqrt{3}$$

Question 6

[Ans: (a) 2.58 (b)(i) $A\left(\ln\sqrt{5},0\right)$; $B\left(0,4\right)$ (ii) -10]

(a)

$$2^x + 4^{x-1} = 15$$
$$2^x + 4^x\left(4^{-1}\right) = 15$$
$$2^x + (2^{2x})\left(\frac{1}{4}\right) = 15$$

Let $u = 2^x$.

$$u + \frac{1}{4}u^2 = 15$$
$$4u + u^2 = 60$$
$$u^2 + 4u - 60 = 0$$
$$(u-6)(u+10) = 0$$
$$u = 6, \quad u = -10$$

$2^x = 6, \quad 2^x = -10$ (undefined, $\because 2^x > 0$)

$$x\ln 2 = \ln 6$$

$x \approx 2.58$ (3 sf)

(b)(i)

At $A, y = 0$

$$0 = 5 - \mathrm{e}^{2x}$$
$$\mathrm{e}^{2x} = 5$$
$$\mathrm{e}^{2x} = 5$$
$$2x = \ln 5$$
$$x = \frac{1}{2}\ln 5$$
$$= \ln\sqrt{5}$$
$$A\left(\ln\sqrt{5},0\right)$$

At $B, x = 0$

$$y = 5 - \mathrm{e}^0 = 4$$
$$B\left(0,4\right)$$

(b)(ii)

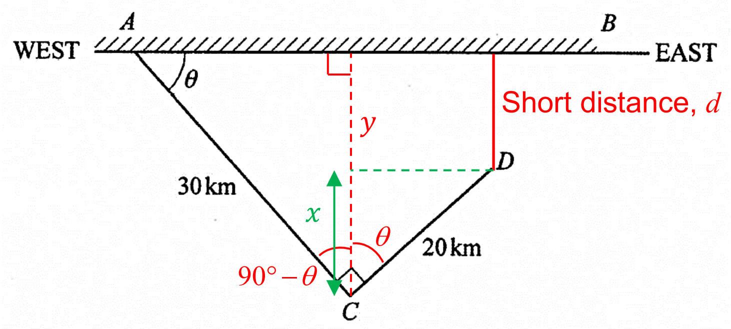
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2\,\mathrm{e}^{2x}$$

At A, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2\,\mathrm{e}^{2\ln\sqrt{5}} = -2(5) = -10$

Question 7

[Ans: (a) Show (b) $\sqrt{1300}\sin(\theta - 33.7°)$ (c) $49.8°$]

(a)



$\sin\theta = \frac{y}{30} \quad \Rightarrow y = 30\sin\theta$

$\cos\theta = \frac{x}{20} \quad \Rightarrow x = 20\cos\theta$

$d = 30\sin\theta - 20\cos\theta$

(b) Let $R\sin(\theta - \alpha) = 30\sin\theta - 20\cos\theta$

$R\sin\theta\cos\alpha - R\cos\theta\sin\alpha = 30\sin\theta - 20\cos\theta$

$R\cos\alpha = 30.....(1)$

$R\sin\alpha = 20.....(2)$

$(1)^2 + (2)^2$:

$R^2\cos^2\alpha + R^2\sin^2\alpha = 30^2 + 20^2$

$R^2\left(\cos^2\alpha + \sin^2\alpha\right) = 1300$

$R^2 = 1300$

$R = \sqrt{1300}$

$\frac{(2)}{(1)} : \frac{\sin\alpha}{\cos\alpha} = \frac{20}{30}$

$\tan\alpha = \frac{2}{3}$

$\alpha = \tan^{-1}\frac{2}{3} \approx 33.69°$

$\therefore 30\sin\theta - 20\cos\theta = \sqrt{1300}\sin(\theta - 33.7°)$ (Angles in degree correct to 1 dp)

(c)

$\sqrt{1300}\sin(\theta - 33.69°) = 10$

$\sin(\theta - 33.69°) = \frac{10}{\sqrt{1300}}$

$\theta - 33.69° = \sin^{-1}\frac{10}{\sqrt{1300}}$

$\theta - 33.69° = 16.10°$

$\theta \approx 49.8°$ (1 dp)

Question 8

[Ans: (a) Show (b) 0.5 units per second]

(a) $y=(x-3)(2x+1)^{\frac{1}{2}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=(1)(2x+1)^{\frac{1}{2}}+\frac{1}{2}(2x+1)^{-\frac{1}{2}}(2)(x-3)$$

$$=(2x+1)^{\frac{1}{2}}+(2x+1)^{-\frac{1}{2}}(x-3)$$

$$=(2x+1)^{-\frac{1}{2}}\left[(2x+1)^{1}+(x-3)\right]$$

$$=(2x+1)^{-\frac{1}{2}}(3x-2)$$

$$=\frac{3x-2}{\sqrt{2x+1}}$$

(b) $\frac{\mathrm{d}x}{\mathrm{d}t}=0.15,\quad \frac{\mathrm{d}y}{\mathrm{d}t}=?\quad ,x=4$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3(4)-2}{\sqrt{2(4)+1}}=\frac{10}{3}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$=\frac{10}{3}\times 0.15$$

$$=0.5$$

Question 8 (continuation)

[Ans: (c) $\frac{10}{3}$]

(c) $\int_0^4 \frac{3x-2}{\sqrt{2x+1}}\,dx = \left[(x-3)(2x+1)^{\frac{1}{2}}\right]_0^4$

$$\int_0^4 \frac{3x}{\sqrt{2x+1}}\,dx - \int_0^4 \frac{2}{\sqrt{2x+1}}\,dx = \left[(x-3)(2x+1)^{\frac{1}{2}}\right]_0^4$$

$$\int_0^4 \frac{3x}{\sqrt{2x+1}}\,dx = \left[(x-3)(2x+1)^{\frac{1}{2}}\right]_0^4 + \int_0^4 2(2x+1)^{-\frac{1}{2}}dx$$

$$\int_0^4 \frac{3x}{\sqrt{2x+1}}\,dx = \left[(x-3)(2x+1)^{\frac{1}{2}}\right]_0^4 + \left[\frac{2(2x+1)^{\frac{1}{2}}}{(2)\left(\frac{1}{2}\right)}\right]_0^4$$

$$\int_0^4 \frac{3x}{\sqrt{2x+1}}\,dx = \left[(x-3)(2x+1)^{\frac{1}{2}}\right]_0^4 + \left[2(2x+1)^{\frac{1}{2}}\right]_0^4$$

$$= \left[(2x+1)^{\frac{1}{2}}(x-3+2)\right]_0^4$$

$$= \left[(2x+1)^{\frac{1}{2}}(x-1)\right]_0^4$$

$$= \left[(9)^{\frac{1}{2}}(3)\right] - \left[(1)^{\frac{1}{2}}(-1)\right]$$

$$= 10$$

$$\int_0^4 \frac{x}{\sqrt{2x+1}}\,dx = \frac{10}{3}$$

Question 9

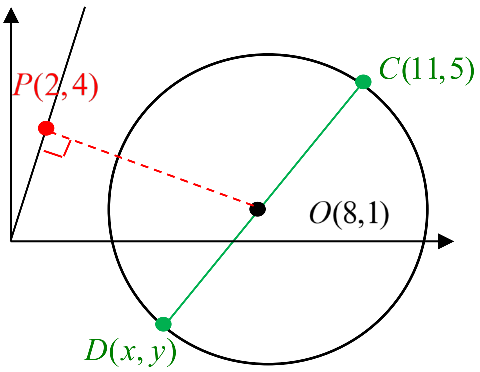
[Ans: (a) Radius $= 5$; Centre $(8,1)$ (b) 6.71 ; Explain (c) $D(5,-3)$]

(a)
$$x^2 + y^2 - 16x - 2y + 40 = 0$$
$$x^2 - 16x + y^2 - 2y + 40 = 0$$
$$(x-8)^2 - 64 + (y-1)^2 - 1 + 40 = 0$$
$$(x-8)^2 + (y-1)^2 = 25$$
$$(x-8)^2 + (y-1)^2 = 5^2$$

Radius $= 5$; Centre $(8,1)$

(b) Line perpendicular to $y = 2x$ and passes through $(8,1)$

$m_1 = 2, m_2 = -\frac{1}{2}$



$y - 1 = -\frac{1}{2}(x-8)$

$2y - 2 = -x + 8$

$x = 10 - 2y......(1)$

$y = 2x......(2)$

Sub (2) into (1)

$x = 10 - 2(2x)$

$5x = 10$

$x = 2$

$y = 2(2) = 4$

$P(2,4)$

Distance $OP = \sqrt{(8-2)^2 + (1-4)^2} = 6.71$

$6.71 > 5$ (Distance $OP >$ radius of circle)

$\therefore$ The circle does not intersect the line $y = 2x$.

(c) O is the midpoint of CD, let $D(x,y)$

$\frac{x+11}{2} = 8, \quad \frac{y+5}{2} = 1$

$x = 5, \quad y = -3$

$\therefore D(5,-3)$

Question 10

[Ans: (a) Nearer to A (b) $a=\frac{9}{2}, b=-4$]

(a) At $A, x=0$

$$y=\frac{9}{3-2(0)}=3$$

$A(0,3)$

$$y=9(3-2x)^{-1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-9(3-2x)^{-2}(-2)$$

$$=\frac{18}{(3-2x)^2}$$

At P, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{18}{\left[3-2(-3)\right]^2}=\frac{2}{9}$

Tangent PB

$$y-1=\frac{2}{9}(x+3)$$

$$y-1=\frac{2}{9}x+\frac{2}{3}$$

$$y=\frac{2}{9}x+\frac{5}{3}$$

$$B\left(0,\frac{5}{3}\right)$$

Distance $AB=3-\frac{5}{3}=1\frac{1}{3}$, $OB=\frac{5}{3}=1\frac{2}{3}$

$\therefore$ Nearer to A

(b) Area under curve

$$\int_{-3}^{0}\frac{9}{3-2x}\,\mathrm{d}x=\left[\frac{9\ln(3-2x)}{-2}\right]_{-3}^{0}$$

$$=-\frac{9}{2}\left[\ln 3-\ln 9\right]$$

$$=-\frac{9}{2}\ln\frac{1}{3}$$

$$=\frac{9}{2}\ln\left(\frac{1}{3}\right)^{-1}$$

$$=\frac{9}{2}\ln 3$$

Area of trapezium $=\frac{1}{2}\left(1+\frac{5}{3}\right)(3)=4$

Shaded region $=\frac{9}{2}\ln 3-4$

$\therefore a=\frac{9}{2}, b=-4$