RAFFLES INSTITUTION
2021 Year 5 H2 Mathematics Promotion Examination
Questions and Solutions with comments

1 A curve C has equation $y=\dfrac{ax+b}{cx-2}$, where a, b and c are constants. It is given that C passes through the points with coordinates $(1,\ 5)$ and $(-8,\ 0.5)$. The curve C is translated 1 unit in the positive x-direction. The new curve passes through the point with coordinates $(0,\ -0.2)$. Find the values of a, b and c. [4]

Solution		**Comments**
1 **[4]**	$y=\dfrac{ax+b}{cx-2}$ Sub $(1,\ 5)$ and $(-8,\ 0.5)$ into equation, $a+b-5c=-10$ ----- (1) $8a-b-4c=1$ ----- (2) After transformation, the translated curve is $y=\dfrac{a(x-1)+b}{c(x-1)-2}$. Substitute $(0,\ -0.2)$, we get $5a-5b+c=-2$ ----- (3) **Alternatively,** Since $(0,\ -0.2)$ lies on the translated curve, then $(-1,-0.2)$ should lie on the original curve. Substitute $(-1,\ -0.2)$ onto the original curve, we get $-0.2=\dfrac{-a+b}{-c-2}$ $5a-5b+c=-2$ -----(3) From GC, $a=2$, $b=3$ and $c=3$, i.e. $y=\dfrac{2x+3}{3x-2}$	Most students were able to substitute the given information in the question and form the 3 equations before using the GC to solve for the values of *a, b* and *c*.

2 The curve C has equation $y^3 = 4 - \frac{xy^2}{2}$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{6y+2x}$. [2]

(ii) Find the equation of the normal to C at the point P where $y = 1$. [3]

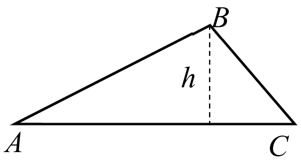
Solution		**Comments**
2(i) **[2]**	Method 1 $y^3 = 4 - \frac{xy^2}{2}$ Differentiate with respect to x: $3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y^2}{2} - xy\frac{\mathrm{d}y}{\mathrm{d}x}$ $\left(3y^2 + xy\right)\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y^2}{2}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y^2}{6y^2+2xy} = -\frac{y}{6y+2x}$ (shown) Method 2 $y^3 = 4 - \frac{xy^2}{2} \quad \Rightarrow \quad x = \frac{8}{y^2} - 2y$ ------(1) Differentiate with respect to y: $\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{16}{y^3} - 2 = -\frac{16+2y^3}{y^3}$ Then, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y^3}{16+2y^3} = -\frac{y}{\frac{16}{y^2}+2y}$ $= -\frac{y}{2x+4y+2y}$, using (1) $= -\frac{y}{6y+2x}$ (shown) Method 3 $y = \left(4 - \frac{xy^2}{2}\right)^{\frac{1}{3}}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3\left(4 - \frac{xy^2}{2}\right)^{\frac{2}{3}}}\left(-xy\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y^2}{2}\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{3y^2}\left(xy\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y^2}{2}\right)$	Students should use implicit differentiation to solve part (i).

	$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{x}{3y}\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{6}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{6}\times\frac{1}{1+\frac{x}{3y}}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{y}{6y+2x}$ (shown)	
(ii) **[3]**	At $y=1,\ \ x=6$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{18}$ Gradient of normal = 18 Hence equation of normal $y-1=18(x-6)$ $y=18x-107$	Very standard question and most students are able to get this correct.

3 The points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

(a) Show that the area of triangle ABC is $\frac{1}{2}\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|$. Hence show that the shortest distance from B to AC is $\frac{\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|}{\left|\mathbf{c}-\mathbf{a}\right|}$. [4]

(b) Given that $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors such that $\left|\mathbf{a}-\mathbf{b}\right|=\left|\mathbf{a}+\mathbf{b}\right|$, find the value of $\mathbf{a}\cdot\mathbf{b}$. [2]

Solution		**Comments**
3(a) **[4]**	area of triangle ABC $= \frac{1}{2}\left|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\right|$ $= \frac{1}{2}\left|(\mathbf{b}-\mathbf{a})\times\mathbf{c}-(\mathbf{b}-\mathbf{a})\times\mathbf{a}\right|$ $= \frac{1}{2}\left|\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}-\mathbf{b}\times\mathbf{a}+\mathbf{a}\times\mathbf{a}\right|$ $= \frac{1}{2}\left|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}+\mathbf{0}\right|$ $= \frac{1}{2}\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|$ (shown) Let shortest distance from B to AC be h (which is also the perpendicular distance from B to AC). $AC=\left|\overrightarrow{AC}\right|=\left|\mathbf{c}-\mathbf{a}\right|$  area of triangle $ABC = \frac{1}{2}(AC)(h) = \frac{1}{2}\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|$ Thus $h=\frac{\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|}{\left|\mathbf{c}-\mathbf{a}\right|}$. (shown)	As this is a "show" question, students need to pen down more working to get full credit for the first part. The second part of (a) is a 'hence' question. Students need to use the result from the earlier part to show the shortest distance from B to AC.
(b) **[2]**	$\left|\mathbf{a}-\mathbf{b}\right|=\left|\mathbf{a}+\mathbf{b}\right|$ $(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})$ $\left|\mathbf{a}\right|^2-2\mathbf{a}\cdot\mathbf{b}+\left|\mathbf{b}\right|^2=\left|\mathbf{a}\right|^2+2\mathbf{a}\cdot\mathbf{b}+\left|\mathbf{b}\right|^2$ $\mathbf{a}\cdot\mathbf{b}=0$ *Alternative* Consider a parallelogram $OACB$ with $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$. Then the lengths of its diagonals are given by $AB=\left|\mathbf{a}-\mathbf{b}\right|$ and $OC=\left|\mathbf{a}+\mathbf{b}\right|$. If $\left|\mathbf{a}-\mathbf{b}\right|=\left|\mathbf{a}+\mathbf{b}\right|$, then $OACB$ forms a rectangle and thus $\mathbf{a}\cdot\mathbf{b}=0$.	Students need to apply the properties of the dot product correctly before any credit is given.

4 A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_n = \sum_{r=1}^{n}(2r+n+1)$. Another sequence $v_1, v_2, v_3, \ldots$ is given by $v_n = \frac{2}{u_n}$, where $n \in \mathbb{Z}^+$.

(i) Find u_n in terms of *n*. [2]

(ii) Show that $v_n = \frac{1}{n} - \frac{1}{n+1}$. [1]

(iii) Describe the behaviour of the sequence $v_1, v_2, v_3, \ldots$. [1]

(iv) Find the sum, S_N, of the first *N* terms of the sequence $v_1, v_2, v_3, \ldots$. [2]

(v) Give a reason why the series S_N converges, and write down the value of the sum to infinity. [2]

Solution		**Comments**
4(i) **[2]**	$u_n = \sum_{r=1}^{n}(2r+n+1) = \frac{n}{2}\left[n+3+3n+1\right] = 2n(n+1)$	u_n is the sum of *n* terms of an AP: first term $n+3$, last term $3n+1$.
(ii) **[1]**	$v_n = \frac{2}{u_n} = \frac{1}{n(n+1)} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ (shown)	Show working clearly as it is a “show” question.
(iii) **[1]**	The sequence $v_1, v_2, v_3, \ldots$ decreases and converges to zero as $\frac{1}{n} \to 0$, and $\frac{1}{n+1} \to 0$.	Question is asking about the sequence, not the series.
(iv) **[2]**	$S_N = \sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+1}\right)$ $= [\ 1 - \cancel{\frac{1}{2}}$ $+\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$ $+\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$ $+\cancel{\ldots}$ $+\ldots$ $+\cancel{\frac{1}{N-1}} - \cancel{\frac{1}{N}}$ $+\cancel{\frac{1}{N}} - \frac{1}{N+1}]$ $= 1 - \frac{1}{N+1}$	Question is asking for S_N, not S_n.
(v) **[2]**	As $N \to \infty$, $S_N = \sum_{n=1}^{N} v_n = 1 - \frac{1}{N+1} \to 1$, since $\frac{1}{N+1} \to 0$. Thus, the series S_N converges. Sum to infinity of the series = 1	Answer the question, i.e. state explicitly the sum to infinity.

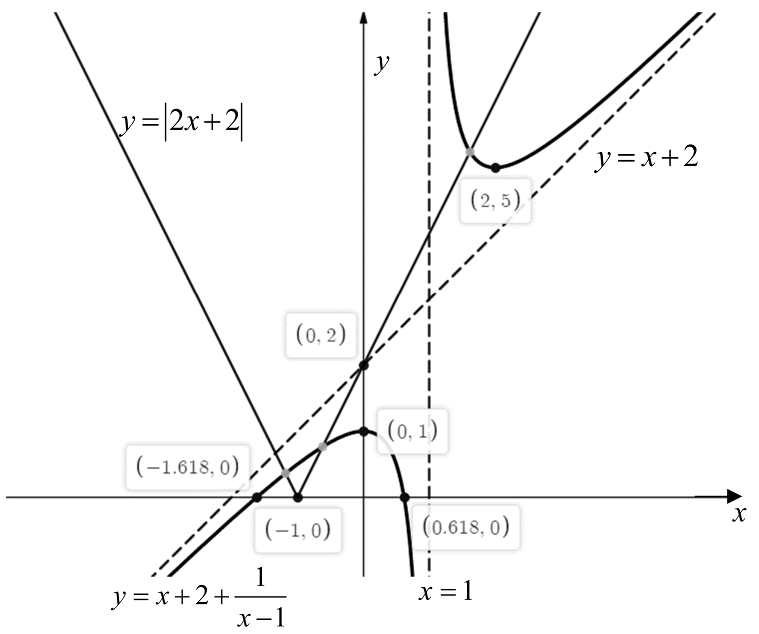
5 **(i)** Using standard series from the List of Formulae (MF26), expand $\dfrac{\cos 3x}{4-x}$ as far as the term in x^3. Give the coefficients as exact fractions in their simplest form. [4]

(ii) It is given that the third and fourth terms found in part **(i)** are equal to the third and fourth terms in the series expansion of $\left(a+bx\right)^5$, in ascending powers of x, respectively. Find the values of the constants a and b. [4]

Solution		**Comments**
5(i) **[4]**	$\left(4-x\right)^{-1}\cos 3x$ $=4^{-1}\left(1-\dfrac{1}{4}x\right)^{-1}\cos 3x$ $\approx\dfrac{1}{4}\left(1-\left(-\dfrac{1}{4}x\right)+\dfrac{-1\left(-2\right)}{2!}\left(-\dfrac{1}{4}x\right)^2+\dfrac{-1\left(-2\right)\left(-3\right)}{3!}\left(-\dfrac{1}{4}x\right)^3\right)\left(1-\dfrac{\left(3x\right)^2}{2!}\right)$ $=\dfrac{1}{4}\left(1+\dfrac{1}{4}x+\dfrac{1}{16}x^2+\dfrac{1}{64}x^3\right)\left(1-\dfrac{9}{2}x^2\right)$ $\approx\dfrac{1}{4}\left(1-\dfrac{9}{2}x^2+\dfrac{1}{4}x-\dfrac{9}{8}x^3+\dfrac{1}{16}x^2+\dfrac{1}{64}x^3\right)$ $=\dfrac{1}{4}\left(1+\dfrac{1}{4}x-\dfrac{71}{16}x^2-\dfrac{71}{64}x^3\right)$ $=\dfrac{1}{4}+\dfrac{1}{16}x-\dfrac{71}{64}x^2-\dfrac{71}{256}x^3$	When applying the standard series from MF 26, replace the 'x' in the standard series correctly.
(ii) **[4]**	$\left(a+bx\right)^5=a^5+5a^4bx+10a^3b^2x^2+10a^2b^3x^3+\ldots b^5x^5$ $10a^3b^2=-\dfrac{71}{64}$ -----(1) $\quad 10a^2b^3=-\dfrac{71}{256}$ -----(2) $(1)\div(2),\ \dfrac{a}{b}=\dfrac{256}{64}=4$ $\therefore a=4b$ Substituting into (1) $10\left(4b\right)^3b^2=-\dfrac{71}{64}$ $640b^5=-\dfrac{71}{64}$ $b^5=-\dfrac{71}{40960}$ $b=-0.280$ (3s.f.) $\quad a=-1.12$(3.s.f.)	

6 **(i)** On the same axes, sketch the graphs of $y = x + 2 + \dfrac{1}{x-1}$ and $y = |2x+2|$, stating the coordinates of any points of intersections with the axes, turning points and the equations of any asymptotes. [5]

(ii) Hence solve the inequality $x + 2 + \dfrac{1}{x-1} > |2x+2|$, giving your answers in exact form. [5]

<table>
<tr><th colspan="3">Solution</th><th>Comments</th></tr>
<tr><td>6(i)
[5]</td><td colspan="2">
</td><td>Take note that the <i>y</i>-intercepts of both oblique asymptote and modulus function are the same.

Maximum point is on the <i>y</i>-axis.

Drawing of graphs can definitely be improved. Many students drew graphs with all the points so close to each other that it is difficult to decipher the handwriting and the details.</td></tr>
<tr><td rowspan="2">(ii)
[5]</td><td>Intersection between
$y = x+2+\dfrac{1}{x-1}$ and $y = 2x+2$
$2x+2 = x+2+\dfrac{1}{x-1}$
$x(x-1) = 1$
$x^2 - x - 1 = 0$
$x = \dfrac{1 \pm \sqrt{5}}{2}$</td><td>Intersection between
$y = x+2+\dfrac{1}{x-1}$ and $y = -2x-2$
$-2x-2 = x+2+\dfrac{1}{x-1}$
$-3x-4 = \dfrac{1}{x-1}$
$(3x+4)(1-x) = 1$
$-3x^2 - x + 3 = 0$
$x = \dfrac{-1 \pm \sqrt{37}}{6}$
From the graph, the intersection occurs at $x < -1$. So, $x = \dfrac{-1-\sqrt{37}}{6}$</td><td rowspan="2">This is a “Hence” question. The expectation is to solve points of intersection with the correct equations. There is no need to solve with the inequality signs.

The question also specified exact solution – means GC answers will not be accepted.</td></tr>
<tr><td colspan="2">From the graphs in (i), the solution is $\dfrac{-1-\sqrt{37}}{6} < x < \dfrac{1-\sqrt{5}}{2}$ or $1 < x < \dfrac{1+\sqrt{5}}{2}$.</td></tr>
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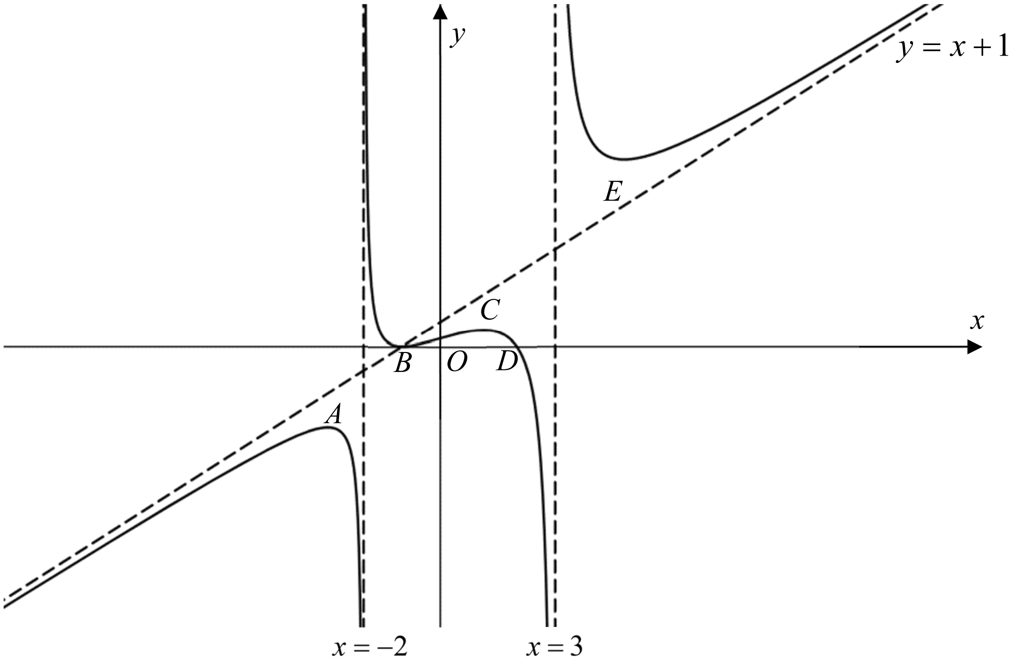
7 **(a)** An arithmetic sequence $a_1,\ a_2,\ a_3,...$ has common difference d, where $d<0$. The sum of the first n terms of the sequence is denoted by S_n. Given that $|a_8|=|a_{13}|$, find the value of n for which S_n is maximum. [4]

(b) The terms $u_1,\ u_2$ and u_3 are three consecutive terms of a geometric progression. It is given that $u_1,\ u_2$ and u_3-32 form an arithmetic progression, and that $u_1,\ u_2-4$ and u_3-32 form another geometric progression. Find the possible values of $u_1,\ u_2$ and u_3. [6]

Solution		**Comments**
7(a) **[4]**	$|a_8|=|a_{13}|$ $|a_1+7d|=|a_1+12d|$ $a_1+7d=a_1+12d$ or $a_1+7d=-a_1-12d$ $7d=12d$ $\quad$ $2a_1+19d=0$ $d=0$ $\quad$ $a_1=-\frac{19}{2}d$ (rejected since $d<0$) **Or** $|a_8|=|a_{13}|$ $(a_1+7d)^2=(a_1+12d)^2$ $(a_1+12d)^2-(a_1+7d)^2=0$ $(2a_1+19d)(5d)=0$ $2a_1+19d_1=0,$ since $d<0$ $a_1=-\frac{19}{2}d$ Thus, $S_n=\frac{n}{2}(2a_1+(n-1)d)=\frac{n}{2}(-19d+nd-d)=\frac{d}{2}n(n-20)$ <u>To find n for max S_n</u> $S_n=\frac{d}{2}n(n-20)$ is a quadratic expression with negative coefficient of n^2 $\left(\frac{d}{2}<0\right)$. When $S_n=0$, $n=0$ or $n=20$. Hence, S_n is maximum at $n=\frac{0+20}{2}=10$.	
	OR $\frac{\mathrm{d}S_n}{\mathrm{d}n}=nd-10d$ When $\frac{\mathrm{d}S_n}{\mathrm{d}n}=0,$ $\quad$ $(n-10)d=0$ $n=10$ since $d<0$ $\frac{\mathrm{d}^2S_n}{\mathrm{d}n^2}=d<0$ Hence, maximum S_n at $n=10$.	When using the differentiation method, it is necessary to use either second derivative test or a complete first derivative test to show that the stationary value is a maximum value.
	OR S_n will keep increasing when each term added is positive, until a maximum, and decrease when the next term added is negative. Consider $a_n>0$, then $-\frac{19}{2}d+(n-1)d>0$ $-\frac{19}{2}+n-1<0$ since $d<0$ $n<10.5$	Other methods should be accompanied with complete and thorough explanations, showing and explaining $a_1>0$, $a_{10}>0$ and

	S_n is maximum when $n=10$ since $a_{10}=-0.5d>0$ and $a_{11}=a_{10}+d=0.5d<0$. **Note that** a_1 has to be positive. If a_1 is negative (and with negative *d*), then a_r will just get smaller and smaller (more and more negative) as *r* increases. Then, it would not be possible that $\lvert a_8\rvert=\lvert a_{13}\rvert$. And so, a_1 has to be positive. In fact, a_8 is positive and a_{13} is negative, and we can say that $a_8=-a_{13}$.	$a_{11}<0$, and drawing link to how this affects S_n in order to obtain full credit.
(b) **[6]**	Let *a* and *r* be the first term and common ratio of the geometric progression. Then, $u_1=a,\ u_2=ar, u_3=ar^2$ $ar-a=ar^2-32-ar \quad ------(*)$ $ar^2-2ar+a=32$ $a(r-1)^2=32$ $(r-1)^2=\dfrac{32}{a} \quad$ -------------- (1) $\dfrac{ar^2-32}{ar-4}=\dfrac{ar-4}{a}\ ------------(**)$ $(ar^2-32)a=a^2r^2-8ar+16$ $-32a=-8ar+16$ $a(8r-32)=16$ $a=\dfrac{2}{r-4} \quad$ --------------- (2) Substituting (2) into (1), $r^2-2r+1=16r-64$ $r^2-18r+65=0$ $r=5 \quad \text{or} \quad r=13$ $a=2 \quad \text{or} \quad a=\dfrac{2}{9}$ $\boxed{\begin{aligned}\left(\frac{2}{a}+3\right)^2&=\frac{32}{a}\\(2+3a)^2&=32a\\9a^2-20a+4&=0\\a=2 \quad \text{or} \quad a&=\frac{2}{9}\end{aligned}}$ Hence $u_1=2,\ u_2=10$ and $u_3=50$ or $u_1=\dfrac{2}{9},\ u_2=\dfrac{26}{9}$ and $u_3=\dfrac{338}{9}$.	One of the key learning points in this question is that students should strive to reduce the number of variables they are dealing with. Those that started with only *a* and *r* have more success than those who dealt with u_1, u_2 and u_3. Also, do read the question carefully, there is no mention of the terms needing to be integers. Solutions that used the guess and check method did not give the second set of answer and the calculators are usually set to show integer values only.

8 The diagram below shows the graph of $y=\mathrm{f}(x)$ with asymptotes $x=-2$, $x=3$ and $y=x+1$. The curve intersects the x-axis at points B and D, and has turning points at points A, B, C and E. The coordinates of A, B, C, D and E are $(-3,-3)$, $(-1,0)$, $\left(\frac{3}{2},\frac{3}{4}\right)$, $(2,0)$ and $(5,8)$ respectively.



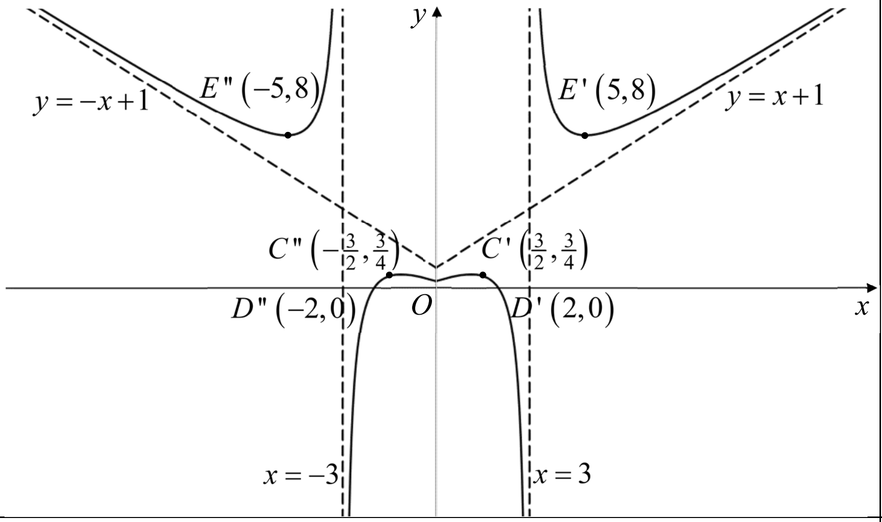
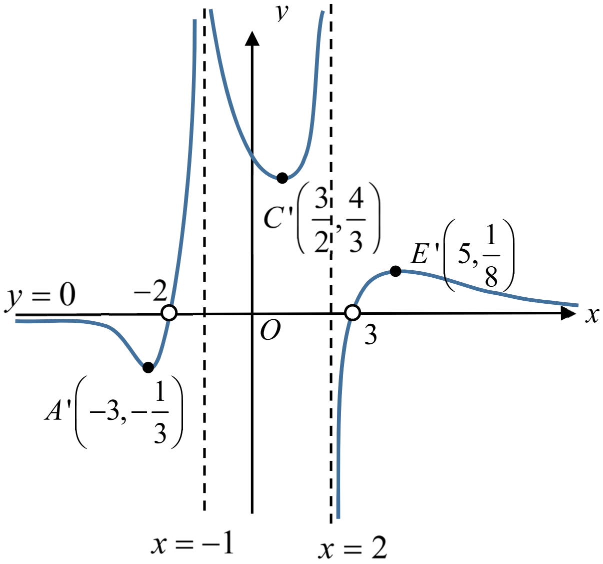
(i) By showing clearly the equations of asymptotes and the coordinates of any turning points and the points where the curve crosses the axes, where possible, sketch, on **separate diagrams**, the graphs of

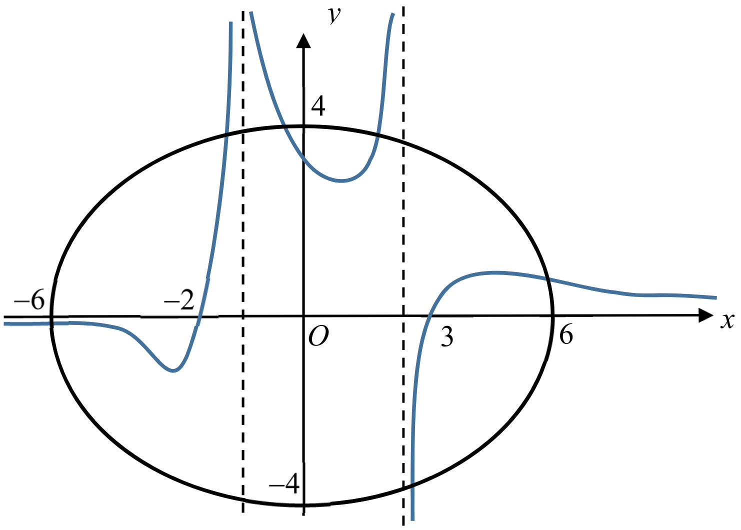
(a) $y=\mathrm{f}(|x|)$, [4]

(b) $y=\dfrac{1}{\mathrm{f}(x)}$. [4]

(ii) By drawing another suitable graph on the same diagram in part **(i)(b)**, determine the number of solutions to the equation $\dfrac{x^2}{36}+\dfrac{1}{16[\mathrm{f}(x)]^2}=1$. [2]

Solution	Comments

8(i) **(a)** **[4]**		Note that there is a sharp point at the *y*-intercept, and not a stationary point.
(b) **[4]**		The graph should be drawn such that it approaches the *x*-axis as $x\to\infty$ and $x\to-\infty$. Since the point *C* is lower than the point *E*, *C'* should be higher than *E'*
(ii) **[2]**	$\frac{x^2}{36}+\frac{1}{\left[\mathrm{f}(x)\right]^2 16}=1$ By drawing $\frac{x^2}{36}+\frac{y^2}{16}=1$ and $y=\frac{1}{\mathrm{f}(x)}$, number of solution is 6	Since we're drawing on the diagram for the graph of $y=\frac{1}{\mathrm{f}(x)}$, we should substitute $\frac{1}{\mathrm{f}(x)}$ in the given equation with y to get the equation of the ellipse. It should not be $y=\mathrm{f}(x)$.

		When drawing 2 graphs on the same diagram, we need to consider their relative position to each other. Here, all 3 turning points are inside the ellipse.

9 Distances in this question are in metres.

Harry and Tom's model airplanes are taking off from the horizontal ground, which is the *x*-*y* plane. Tom's airplane takes off after Harry's. The position of Harry's airplane t seconds after it takes off is given by $\mathbf{r}=(5\mathbf{i}+6\mathbf{j})+t(-4\mathbf{i}+2\mathbf{j}+4\mathbf{k})$. The position of Tom's airplane s seconds after it takes off is given by $\mathbf{r}=(-39\mathbf{i}+44\mathbf{j})+s(4\mathbf{i}-6\mathbf{j}+7\mathbf{k})$.

(i) State the height of Harry's airplane two seconds after it takes off and find its distance travelled in the two seconds. [3]

(ii) Find the acute angle between the path of Harry's airplane and the ground. [2]

(iii) Show that the paths of the airplanes are perpendicular. [1]

(iv) Given that the two airplanes collide, find the coordinates of the point of collision. How long after Harry's airplane takes off does Tom's airplane take off? [3]

(v) Find the cartesian equation of the plane in which both paths of the airplanes lie. [3]

Solution		**Comments**
9(i) **[3]**	When $t=2$, $\mathbf{r}=(5\mathbf{i}+6\mathbf{j})+2(-4\mathbf{i}+2\mathbf{j}+4\mathbf{k})=-3\mathbf{i}+10\mathbf{j}+8\mathbf{k}$ Height of Harry's airplane = 8 m Distance travelled = $\left\lvert\begin{pmatrix}-3\\10\\8\end{pmatrix}-\begin{pmatrix}5\\6\\0\end{pmatrix}\right\rvert=\left\lvert\begin{pmatrix}-8\\4\\8\end{pmatrix}\right\rvert=\sqrt{(-8)^2+4^2+8^2}$ = 12 m	Since the ground is the *x*-*y* plane, the height is given by the *z*-component. Note that the airplane did not start flying from the origin but at point (5, 6, 0).
(ii) **[2]**	Let the angle of takeoff of Harry's airplane from the ground be θ. $\sin\theta=\dfrac{\begin{pmatrix}-4\\2\\4\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}{6}=\dfrac{2}{3}$ $\theta=41.8^\circ$	Vector perpendicular to the *x*-*y* plane is **k**.
(iii) **[1]**	The paths of Harry's airplane and Tom's airplane are parallel to $\begin{pmatrix}-4\\2\\4\end{pmatrix}$ and $\begin{pmatrix}4\\-6\\7\end{pmatrix}$ respectively. Since $\begin{pmatrix}-4\\2\\4\end{pmatrix}\cdot\begin{pmatrix}4\\-6\\7\end{pmatrix}=-16-12+28=0$, the paths of the airplanes are perpendicular.	
(iv) **[3]**	Since the airplanes collide,	

	$\begin{pmatrix}5\\6\\0\end{pmatrix}+t\begin{pmatrix}-4\\2\\4\end{pmatrix}=\begin{pmatrix}-39\\44\\0\end{pmatrix}+s\begin{pmatrix}4\\-6\\7\end{pmatrix}$ for some $s,t\in\mathbb{R}$ $-4t-4s=-44$ --- (1) $2t+6s=38$ --- (2) $4t-7s=0$ --- (3) From the GC, $t = 7$, $s = 4$. Position vector of the point of collision = $\begin{pmatrix}5\\6\\0\end{pmatrix}+7\begin{pmatrix}-4\\2\\4\end{pmatrix}=\begin{pmatrix}-23\\20\\28\end{pmatrix}$ Thus coordinates of the point of collision = $(-23, 20, 28)$. Tom's airplane takes off 3 seconds after Harry's airplane takes off.	Note that question requires the answer to be given in coordinates.
(v) **[3]**	Vector perpendicular to the plane = $\begin{pmatrix}-4\\2\\4\end{pmatrix}\times\begin{pmatrix}4\\-6\\7\end{pmatrix}=\begin{pmatrix}38\\44\\16\end{pmatrix}=2\begin{pmatrix}19\\22\\8\end{pmatrix}$ Equation of the plane containing both paths of the airplanes is $\mathbf{r}\cdot\begin{pmatrix}19\\22\\8\end{pmatrix}=\begin{pmatrix}5\\6\\0\end{pmatrix}\cdot\begin{pmatrix}19\\22\\8\end{pmatrix}$ $\mathbf{r}\cdot\begin{pmatrix}19\\22\\8\end{pmatrix}=227$ Cartesian equation of the plane is $19x+22y+8z=227$.	

10 Functions f and g are defined by

$$\mathrm{f}: x \mapsto 4x - 2k \qquad \text{for } x \in \mathbb{R}, \text{ where } k \text{ is a constant,}$$
$$\mathrm{g}: x \mapsto \frac{9}{2-x} \qquad \text{for } x \in \mathbb{R},\ x \neq 2.$$

(i) Explain why gf does not exist. [1]

(ii) Find the range of values of k for which the equation $\mathrm{fg}(x) = x$ has real roots. [4]

For the rest of the question, let $k = 5$.

(iii) Sketch the graph of $y = \mathrm{fg}(x)$ for $x < 2$. Hence sketch the graph of $y = (\mathrm{fg})^{-1}(x)$ on the same diagram, showing clearly the relationship between the two graphs. [4]

The function h represents the height in metres of an object at time t seconds and is defined for the domain $0 \le t \le b$ by

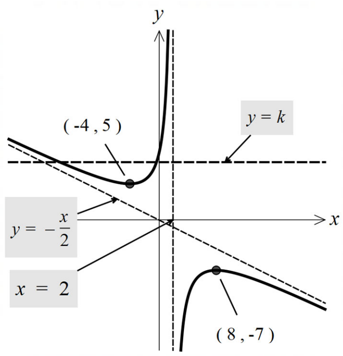
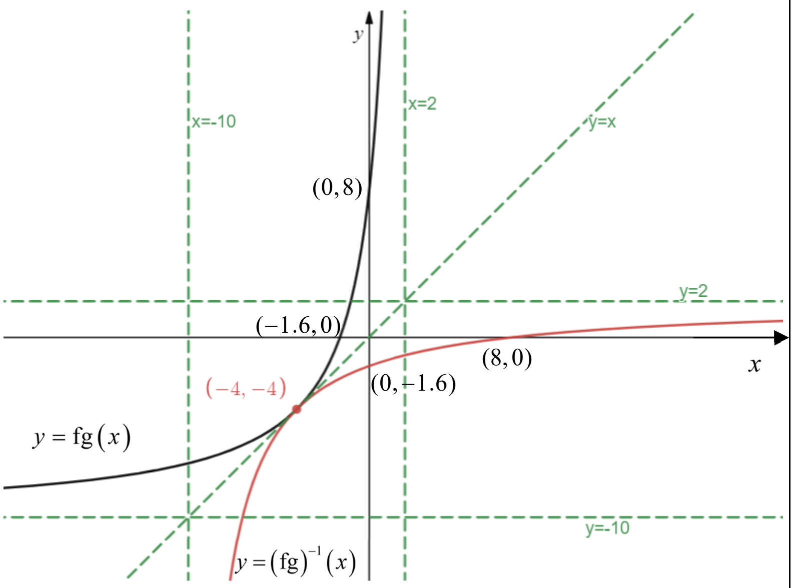
$$\mathrm{h}(t) = \begin{cases} \dfrac{30}{7} + \mathrm{g}(t+9) & \text{for } 0 \le t \le a, \\ 2 - \mathrm{f}(t) & \text{for } a < t \le b, \end{cases}$$

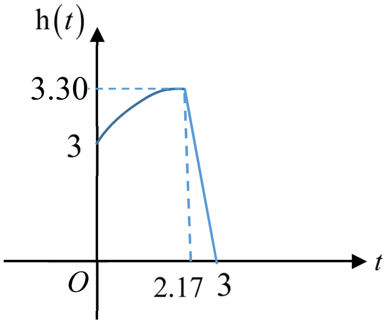
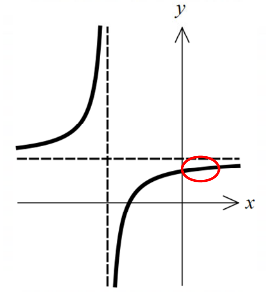
where a and b are constants. At $t = 0$, the object was thrown up from 3 metres above the ground level. When $t = a$, the object started to descend and finally reached the ground at $t = b$.

(iv) Find the values of a and b. [2]

(v) Sketch the graph of $y = \mathrm{h}(t)$ for $0 \le t \le b$. [1]

Solution		**Comments**
10 (i) [1]	$\mathrm{R_f} = (-\infty, \infty)$ $\mathrm{D_g} = (-\infty, \infty) \setminus \{2\}$ Since $\mathrm{R_f} \not\subseteq \mathrm{D_g}$, gf does not exist.	Some only state down the condition and did **not** explicitly give the range and domain, thus, not awarded any marks. Common errors in writing sets: • $\mathrm{R_f} \in \mathbb{R}$ • $\mathrm{R_f} = x \in \mathbb{R}$ • $\mathrm{D_g} = (-\infty, \infty) / \{2\}$ • $\mathrm{D_g} = x \neq 2$
(ii) [4]	$\mathrm{f}(x) = 4x - 2k$; $\mathrm{g}(x) = \dfrac{9}{2-x}$ So, $\mathrm{fg}(x) = 4\left(\dfrac{9}{2-x}\right) - 2k = \dfrac{36}{2-x} - 2k$ **Method 1** $\mathrm{fg}(x) = x$ $\dfrac{36}{2-x} - 2k = x$ $36 - 2k(2-x) = x(2-x)$ $36 - 4k + 2kx = 2x - x^2$ $x^2 + (2k-2)x + 36 - 4k = 0$	There are a few who gave $\mathrm{fg}(x) = \dfrac{9}{2-(4x-2k)}$ when this is actually gf(x). There are quite a few careless mistakes in arriving at this quadratic equation, either the sign is wrong or 36 is mission or -4k is missing.

	For real roots, discriminant ≥ 0 , $(2k-2)^2-4(36-4k)\geq 0$ $k^2+2k-35\geq 0$ $(k+7)(k-5)\geq 0$ Therefore $k\leq -7$ or $k\geq 5$ **<u>Method 2</u>** $\mathrm{fg}(x)=x$ $\frac{36}{2-x}-2k=x$ $-\frac{1}{2}x+\frac{18}{2-x}=k$  By considering the intersection between $y=-\frac{1}{2}x+\frac{18}{2-x}$ and $y=k$, $\mathrm{fg}(x)=x$ has real roots is equivalent to the graphs of $y=-\frac{1}{2}x+\frac{18}{2-x}$ and $y=k$ intersect. The range of the graph of $y=-\frac{1}{2}x+\frac{18}{2-x}$ is $(-\infty,-7])\cup[5,\infty)$. Thus, the range of k is $(-\infty,-7]\cup[5,\infty)$.	Note that if the quadratic equation has 2 real and different roots, then discriminant > 0. If the quadratic equation has 2 real and equal roots, then discriminant = 0. Thus, if the quadratic equation has real roots, then discriminant ≥ 0 .
(iii) **[4]**	$\mathrm{fg}(x)=4\left(\frac{9}{2-x}\right)-10=-10+\frac{36}{2-x}=\frac{16+10x}{2-x}$ $\Rightarrow y=\mathrm{fg}(x)$ has asymptotes $x=2$ and $y=-10$. 	**1)** $y=\mathrm{fg}(x)$ is a rational function of the form linear/linear, and so, it has vertical and horizontal asymptotes. However, many did not give the equation of the horizontal asymptote. **2)** Although the domain of fg is $(\infty,2)$, quite a few thought that the domain of $(\mathrm{fg})^{-1}$ is also $(\infty,2)$. However, recall that $\mathrm{D}_{(\mathrm{fg})^{-1}}=\mathrm{R}_{\mathrm{fg}}=(-10,\infty)$. **3)** When $k=5$, $\mathrm{fg}(x)=x$ has equal roots. Thus, the line $y=x$ is a tangent to the graph of $y=\mathrm{fg}(x)$. **4)** It is crucial that when we draw the graphs of $y=$ function and $y=\text{function}^{-1}$ on the same diagram, the x and y scale should be the same. **5)** The lines $y=x$, $y=2$ and $x=2$ should be concurrent. Likewise $y=x$, $y=-10$ and $x=-10$.

(iv) **[2]**	$\mathrm{h}(t)=\begin{cases}\frac{30}{7}+\frac{9}{2-(t+9)} & \text{for } 0\le t\le a\\ 2-(4t-10) & \text{for } a<t\le b\end{cases}$ When $t=a$, $\frac{30}{7}+\frac{9}{2-(a+9)}=2-(4a-10)$ $-\frac{9}{7+a}=\frac{54}{7}-4a$ Using GC, $a=2.1738=2.17$ (3s.f.) At $t=b$, $h(t)=0$. Thus, $2-(4b-10)=0 \Rightarrow b=3$.	As the height h is a continuous function, $\frac{30}{7}+\mathrm{g}(t+9)$ at $t=a$ is equal to $2-\mathrm{f}(t)$ at $t=a$.
(v) **[1]**		Very few are able to draw the correct graph. Note that when $0\le t\le a$, $y=\frac{30}{7}+\mathrm{g}(t+9)=\frac{30}{7}-\frac{9}{t+7}$ and the graph should be concave downwards. It can be seen from the graph ($y=\frac{30}{7}-\frac{9}{x+7}$) below:  Note that when $a<t\le b$, $y=2-\mathrm{f}(t)=12-4t$ is a straight line.