1 **(i)** Show that $\dfrac{\sin\left[(n+1)\theta - n\theta\right]}{\cos n\theta \cos(n+1)\theta} = \tan(n+1)\theta - \tan n\theta.$ [2]

(ii) Hence find, in terms of n and θ, an expression for

$$\sec\theta\sec 2\theta + \sec 2\theta\sec 3\theta + \sec 3\theta\sec 4\theta + ... + \sec n\theta\sec(n+1)\theta, \text{ where } n \in \mathbb{Z}^+.$$ [3]

2 The number of bacteria (in millions) in culture A at the start of the n^{th} day is denoted by u_n, for $n \in \mathbb{Z}^+$. After the start of each day, a researcher subjects culture A to high temperatures that kill 60% of the existing bacteria. At the end of each day, 3 million new bacteria are produced in culture A. There were 5 million bacteria at the start of the first day.

(i) Write down a sequence in the form $u_{n+1} = au_n + b$, where a and b are constants. [1]

(ii) Describe the behavior of the number of bacteria in culture A in the long run. [1]

In culture B, the number of bacteria (in millions) at the start of the n^{th} day is denoted by

$$v_n = \frac{pn}{n^2 + qn + r}, \text{ for } n \in \mathbb{Z}^+ \text{ and where } p, q, r \text{ are constants.}$$

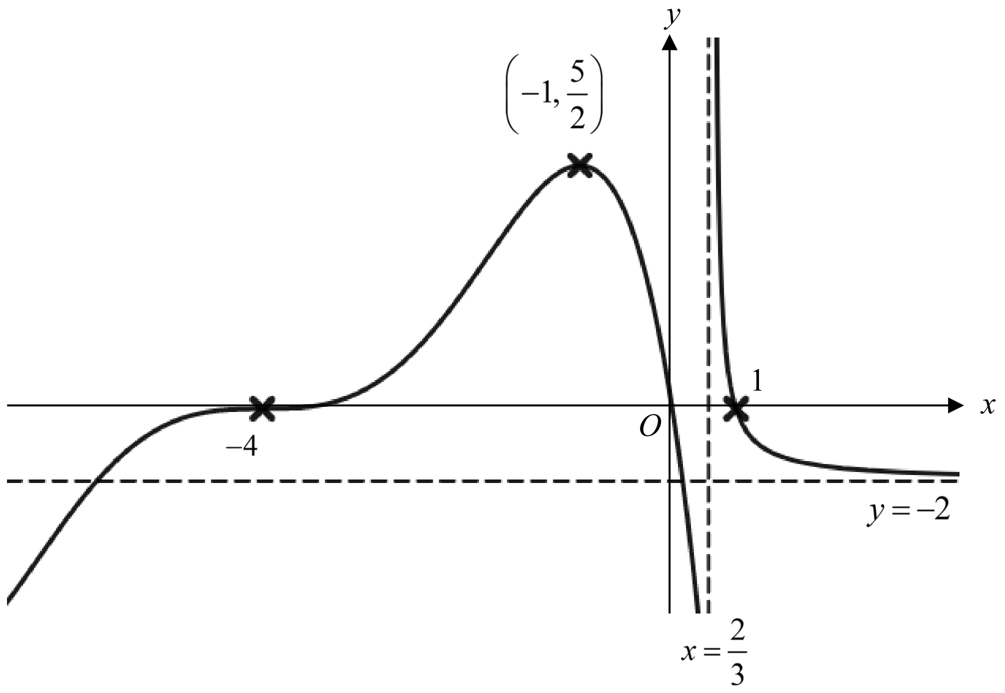
The researcher started the experiment for culture B on 1 April and collected the following data:

At the start of	Number of bacteria (in millions) present in culture B
1 April	2
2 April	2.4
4 April	1.6

(iii) Find the value of p, q and r. [3]

(iv) On which date will the researcher first record the number of bacteria in culture B to be below half a million? [1]

3



The diagram shows the curve with equation $y=\mathrm{f}(x)$, for $x \in \mathbb{R}, x \neq \frac{2}{3}$. The curve crosses the axes at $x=-4$, $x=1$ and the origin, and has asymptotes with equations $x=\frac{2}{3}$ and $y=-2$. The curve has a stationary point of inflexion at $x=-4$ and a turning point with coordinates $\left(-1, \frac{5}{2}\right)$.

(i) Sketch the curve $y=\frac{1}{\mathrm{f}(x)}$, labelling any axial intercepts and coordinates of turning points, and the equations of any asymptotes. [3]

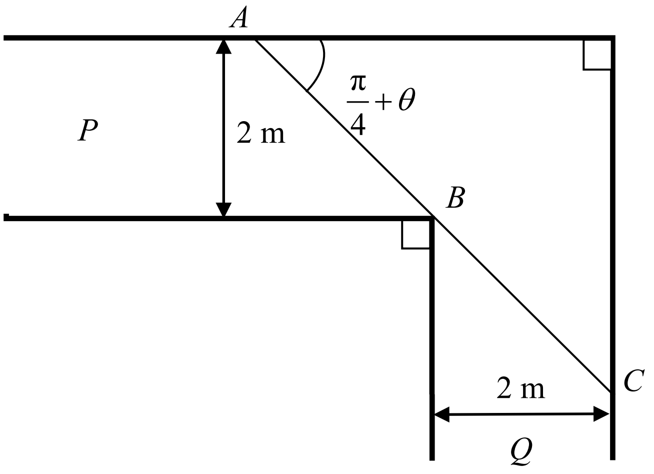
(ii) Sketch the curve $y=\mathrm{f}'(x)$, labelling any x-intercepts and the equations of any asymptotes. [3]

4 **(i)** Differentiate $\mathrm{e}^{\cos 2x}$ with respect to x. [1]

(ii) Find $\int \mathrm{e}^{\cos 2x} \sin 4x \, \mathrm{d}x$. [3]

(iii) Hence find $\int \mathrm{e}^{\cos 2x}(\cos 3x \sin x) \, \mathrm{d}x$. [3]

5



Two straight corridors, P and Q, each of width 2 m, meet at right angles. A banner is hung across the ceiling of the corridors using a taut string such that the string is parallel to the ground and always touches the inside corner of the wall at point B. The string also touches the outer walls at variable points A and C respectively. In the position shown in the diagram, the acute angle between AC and the wall of corridor P is $\frac{\pi}{4}+\theta$, where θ is a sufficiently small angle.

(i) Show that $AC = 2\left[\dfrac{1}{\sin\left(\frac{\pi}{4}+\theta\right)}+\dfrac{1}{\cos\left(\frac{\pi}{4}+\theta\right)}\right]$. [2]

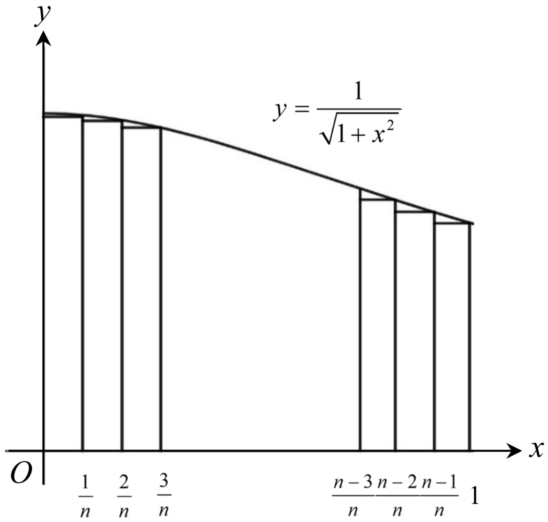
(ii) Hence show that

$$AC \approx r + s\theta^2$$

where r and s are constants to be determined. [5]

6 **(i)** Using the substitution $x = \tan\theta$, find the exact value of $\int_0^1 \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x$. [4]

(ii) The graph of $y = \frac{1}{\sqrt{1+x^2}}$, for $0 \le x \le 1$, is shown in the diagram. Rectangles, each of width $\frac{1}{n}$, are drawn under the curve.



Show that the total area A of all n rectangles is given by

$$A = \frac{1}{\sqrt{n^2+1^2}} + \frac{1}{\sqrt{n^2+2^2}} + \frac{1}{\sqrt{n^2+3^2}} + \ldots + \frac{1}{\sqrt{n^2+(n-1)^2}} + \frac{1}{\sqrt{2n^2}}.$$

State the limit of A as $n \to \infty$. [3]

7 A sequence of positive numbers $u_1, u_2, u_3, \ldots$ is a strictly increasing arithmetic progression. It is given that the first term is a and the ninth term is b.

(i) Find u_3 in terms of a and b and show that $u_3 + u_5 + u_7 = \frac{3}{2}(b+a)$. [3]

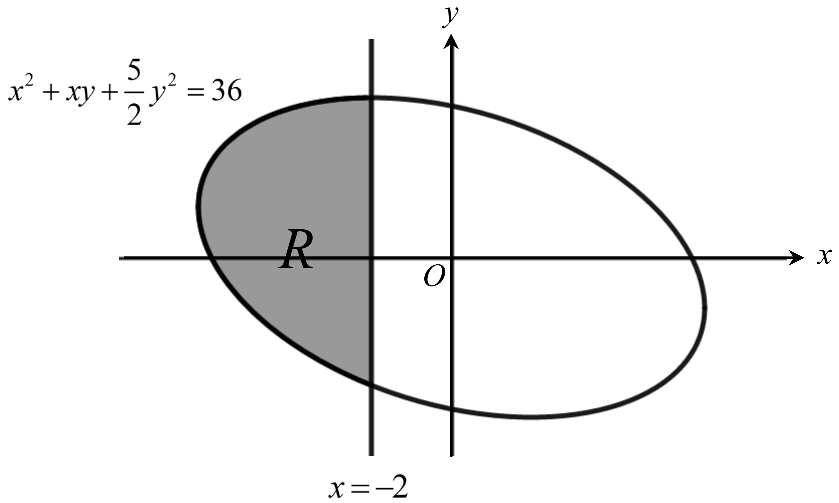
(ii) Given also that a, u_3 and b are consecutive terms of a geometric progression, express b in terms of a. [3]

(iii) Hence, determine if a sequence that consists of consecutive terms $\ln(u_3), \ln(u_5)$ and $\ln(u_7)$ is an arithmetic progression. [2]

8 The curve C has equation $x^2 + xy + ay^2 = 36$, where a is a constant such that $a > \frac{1}{4}$.

(i) Find the x-coordinates of the points on C where the normal is parallel to the y-axis, leaving your answers in terms of a. [4]

(ii) For $a = \frac{5}{2}$, the region R is bounded by C and the line $x = -2$ as shown in the diagram. It is also given that all points in the region R are such that $x \le -\frac{y}{2}$.



Find the volume formed when R is rotated completely about the y-axis, leaving your answers correct to 2 decimal places. [4]

9

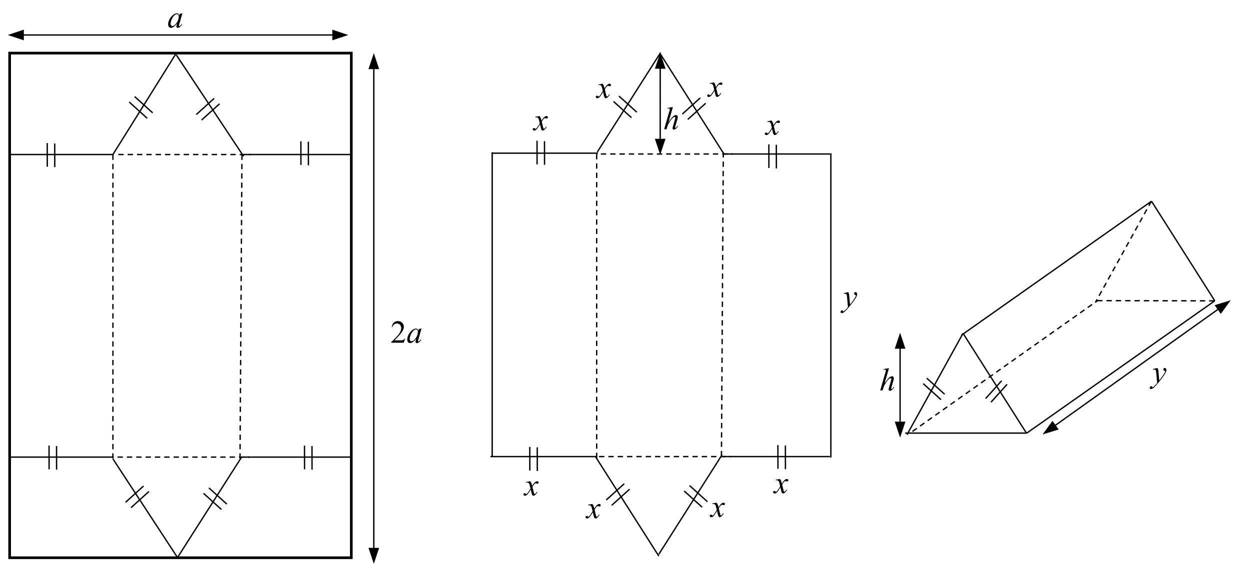


Figure 1 **Figure 2** **Figure 3**

Figure 1 shows a piece of card in the shape of a rectangle with sides a metres and $2a$ metres. A trapezium is cut from each corner, to give the shape shown in **Figure 2** which consists of two identical isosceles triangles and three rectangles. For the triangles, the two equal sides are of length x metres each and the height is h metres. The remaining card shown in **Figure 2** is then folded along the dotted lines to form a closed triangular prism with height y metres as shown in **Figure 3**. The volume of the closed triangular prism is denoted by V.

(i) Find a formula for x in terms of h and a. Hence show that the value of h that gives a stationary value of V satisfies the equation $-16h^3+12ah^2+2a^2h-a^3=0$. [6]

(ii) Suppose $a=5$. Find the value of h that gives a stationary value of V, and explain why there is only one answer. Hence prove that this stationary value of V is a maximum. [4]

10 Antibiotics are used to treat bacterial infections. The rate at which the amount of antibiotics in a patient's body decays is proportional to the amount of antibiotics in the patient's body, x, at any time t in hours. It is given that an initial dose of antibiotics with amount x_0 is administered to a patient. After 6 hours, the amount of antibiotics in the patient's body is $\dfrac{x_0}{1000}$.

(i) Write down a differential equation relating x and t. [1]

(ii) Solve this differential equation to find an expression for x in the form $\dfrac{x_0}{P^t}$, where P is an exact constant to be determined. Hence find the time taken for the amount of antibiotics in the patient's body to reach 25% of the initial dose. [6]

As the amount of antibiotics in the patient's body decays with time, a pharmacist recommends administering the antibiotics every T hours with a dosage of x_0, for an extended period of time.

(iii) State the amount of antibiotics in the patient's body immediately after the second dose. Hence show that the amount of antibiotics in the patient's body at any time, t, after the second dose and before the third dose is $x_0\left(10^{\frac{1}{2}(T-t)}+10^{\left(-\frac{t}{2}\right)}\right)$, for $T \le t < 2T$. [3]

11 **(i)** Sketch the curve with equation $y=\frac{1}{2}+\frac{1}{|x-2|-3}$, stating the equations of the asymptotes.

Hence solve the inequality $\frac{1}{2}+\frac{1}{|x-2|-3}\geq\frac{1}{x}-\frac{1}{3}$. [4]

The functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{1}{2}+\frac{1}{|x-2|-3},\ x\in\mathbb{R},\ -1<x<1,$$

$$\mathrm{g}: x \mapsto \sin\left(\frac{\pi x}{c}\right),\ x\in\mathbb{R},\ \frac{5c}{3}\leq x\leq\frac{14c}{5} \text{ where } c\in\mathbb{R}^{+}.$$

(ii) Find f^{-1} and state its domain. [3]

(iii) Find the exact range of g. [2]

The function h is given by

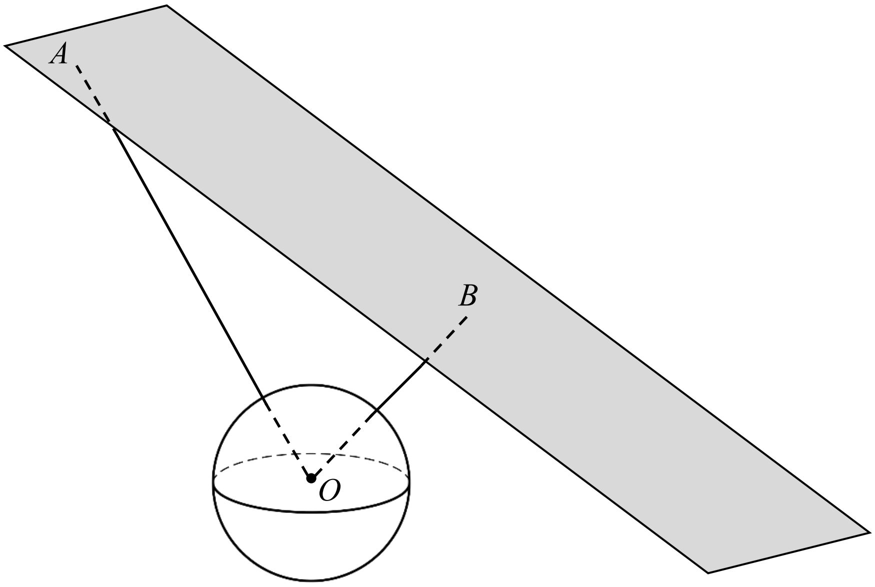
$$\mathrm{h}(x)=\mathrm{g}(x),\ x\in\mathbb{R},\ \frac{3c}{2}<x<\frac{5c}{2}$$

where $c\in\mathbb{R}^{+}$.

(iv) Find $(\mathrm{fh})^{-1}\left(-\frac{1}{2}\right)$ in terms of c. [3]

12 **(a)** The lines l_1 has equation $\mathbf{r} = 10\mathbf{i} + 8\mathbf{j} + 8\mathbf{k} + \lambda(\mathbf{i} + 14\mathbf{j} + h\mathbf{k})$, where λ is a parameter and h is a constant. Another line l_2 has equation $\mathbf{r} = s\mathbf{i} - 10\mathbf{j} + 12\mathbf{k} + \mu(2\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})$, where μ is a parameter and s is a constant. Given that l_1 and l_2 are skew lines that are perpendicular, find the possible values of h and s. [4]

(b)



In an exhibition hall, an advertisement ball in the shape of a sphere with radius 1 unit is suspended from the roof of a building using hanging cords. Points (x, y, z) are defined relative to the centre of the ball at $(0,0,0)$, where units are in metres. Cords connecting the ball to the roof are straight lines and the thickness of the cords can be neglected.

The roof can be modelled by a plane with equation $6x + 8z = 25$. Cord *OA* starts at the centre of the ball and the coordinate of *A* is $(-2.5, 0, 5)$. Cord *OB* also starts at the centre of the ball and it is the shortest possible cord from the centre of the ball to the roof.

(i) Find the coordinates of *B* and hence find the shortest distance between the surface of the advertisement ball and the roof. [3]

To further secure the suspended advertisement ball, a third hanging cord *OD* is added such that cord *OD* is the reflection of cord *OA* in cord *OB*.

(ii) Find an equation of the line representing cord *OD*. [3]

A square LED light panel that is part of a plane is to be installed between the advertisement ball and the roof. The distance between the plane containing the LED light panel and the roof is 0.8 metres. Assume that the thickness of the LED light panel is negligible.

(iii) Find a cartesian equation of the plane which represents the LED light panel. [2]

(iv) It is given further that the square LED light panel has sides of length n metres and its centre passes through cord *OB*. Find the largest possible integer n such that the panel will not touch cord *OA*. [2]