Temasek Junior College
2020 JC 2 Prelim Exam H2 Mathematics Paper 2

1 A sequence of numbers $u_1, u_2, u_3, \ldots$ has a sum S_n where $S_n = \sum_{r=1}^{n} u_r$. $\boxed{S_1 + S_2 + S_3}$

(i) It is known that the sum of the first three terms of the sequence is 17 and $\sum_{n=1}^{3} S_n = 30$.

Furthermore, the third term of the sequence is twice the first term of the sequence. Find the first three terms of the sequence. [3]

(ii) A constant k is subtracted from each term of the sequence so that the first three terms of the modified sequence are now three consecutive terms of a geometric progression with a common ratio greater than 1.
Find the value of k and the common ratio of the geometric progression. [3]

[Solution]

(i) $u_1 + u_2 + u_3 = 17$ --- (1)

$S_1 + S_2 + S_3 = 30$

$u_1 + (u_1 + u_2) + (u_1 + u_2 + u_3) = 30$

$3u_1 + 2u_2 + u_3 = 30$ --- (2)

$u_3 = 2u_1 \Rightarrow 2u_1 - u_3 = 0$ --- (3)

Only a handful of students used the GC to solve the system of 3 linear equations. The GC can help save time and is less likely to have

From GC, first 3 terms of sequence are $u_1 = 4$, $u_2 = 5$ and $u_3 = 8$

(ii) After subtracting k, the three terms are consecutive terms of a GP

$\Rightarrow \frac{8-k}{5-k} = \frac{5-k}{4-k}$

$\Rightarrow 32 - 12k + k^2 = 25 - 10k + k^2$

$\Rightarrow k = \frac{7}{2}$

$\Rightarrow r = \frac{8-\frac{7}{2}}{5-\frac{7}{2}} = 3$ (since $r > 1$).

2 The functions f and g are defined by

$$\mathrm{f} : x \mapsto 5-(x-2)^2, x \in \mathbb{R},\ x \le 3,$$

$$\mathrm{g} : x \mapsto 3-\mathrm{e}^{-2x},\ x \in \mathbb{R},\ x \le k,$$

where k is a constant.

(i) Sketch the graph of $y = \mathrm{f}(x)$, indicating clearly the coordinates of the turning point, the end-point and the points where the curve meets the axes. Explain why f^{-1} does not exist. [3]

(ii) Explain why the composite function fg exists for all real values of k. [2]

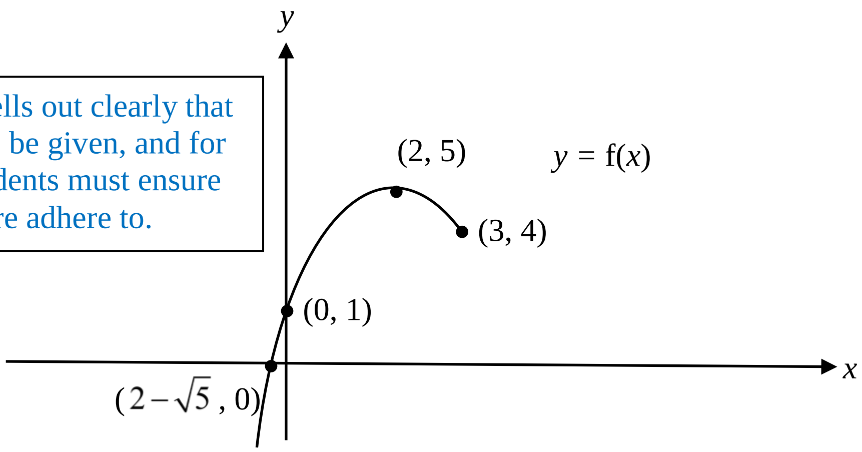
(iii) Find the maximum value of k for which fg is one-one. [3]

(iv) For the value of k found in **(iii)**, find the value of x for which $\mathrm{fg}(x) = (\mathrm{fg})^{-1}(x)$. [2]

[Solution]

(i)

The questions spells out clearly that coordinates are to be given, and for which points, students must ensure on these details are adhere to.



The line $y = 4$ cuts the graph of $y = \mathrm{f}(x)$ twice. Hence f is not 1-1 and f^{-1} does not exist.

Do not confuse the x- and y-coordinates which result in identifying, for example, $y = 2$, $y = 2.5$ or $y = 3$ as lines which intersect $y = \mathrm{f}(x)$ more than once.

(ii) $R_g = \left(-\infty, 3-\mathrm{e}^{-2k}\right]$

For all $k \in \mathbb{R}$, $\mathrm{e}^{-2k} > 0$ and so $3 - \mathrm{e}^{-2k} < 3$, hence $R_g \subset D_f$

$\therefore$ fg exists for all real values of k.

This is equivalent to a "show" problem, as the outcome (fg exists) is given to you. So all steps must be shown, including a mention that $3 - \mathrm{e}^{-2k} < 3$ so that $R_g \subset D_f$

Note: Because $D_g = (-\infty, k]$, R_g will always be of the form $(-\infty, m]$ for some $m \in \mathbb{R}$. In particular, it is actually not possible to get $R_g = (-\infty, 3)$ no matter how large k is. There are many students who correctly stated that as $k \to \infty$, $\mathrm{g}(x) \to 3$. However, this does not imply that $R_g = (-\infty, 3)$.

(iii) $D_{fg} = D_g = (-\infty, k]$

From (i), for fg to be 1-1, we need $R_g \subseteq (-\infty, 2]$

Hence $3 - e^{-2k} \leq 2$

$\Rightarrow k \leq 0$;

Therefore, the maximum value of k is 0.

R_g need not be equal to $(-\infty, 2]$ for fg to be one-one

Starting the solution with an equation instead of an inequality will not show that the obtained value of k is a

Alternative

$D_{fg} = D_g = (-\infty, k]$

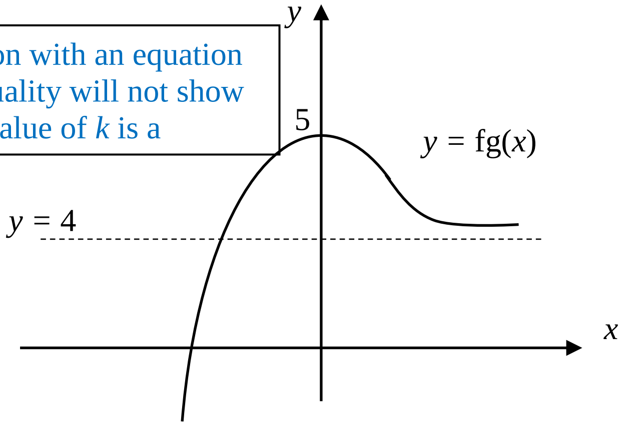
Draw graph of $y = \text{fg}(x)$

$\text{fg}(x) = 5 - (3 - e^{-2x} - 2)^2 = 5 - (1 - e^{-2x})^2$

Since $D_{fg} = (-\infty, k]$, from the graph,

fg is one-one as long as $k \leq 0$.

Hence maximum $k = 0$.



(1)The explanations must

i. highlight that $D_{fg} = (-\infty, k]$

ii. explain the obtained answer was a maximum.

(2) There were a number of answers referencing a maximum point – however, a maximum point does not in any way guarantee a one-one function. Consequently it is not a good approach to try to use the maximum point as a way to approach the question. Looking only at the maximum point also ignores the fact that fg can be one-one without reaching the maximum point, for example, if $D_{fg} = (-\infty, -3]$.

(3) Other misconceptions include: "fg is one-one $\Rightarrow$ f is one-one", or thinking that the k in this question refers to $y = k$, a horizontal line.

(iv) By symmetry, when $\text{fg}(x) = x = (\text{fg})^{-1}(x)$.

$\text{fg}(x) = 5 - (3 - e^{-2x} - 2)^2 = 5 - (1 - e^{-2x})^2 = x$.

From GC, $x = -0.607$ (3sf) since $x \leq 0$.

Students who tried to find $(\text{fg})^{-1}$ generally did not do well for this part. Also, it should be observed that this is a 2-mark part, which should not require extensive working.

3 It is given that $\mathrm{f}(x)=\sin[\ln(1+x)]$, $x\in\mathbb{R}$, $x>-1$.

(i) Show that $(1+x)^2\mathrm{f}''(x)+(1+x)\mathrm{f}'(x)+\mathrm{f}(x)=0$. [2]

(ii) By further differentiation of the result in **(i)**, find the first three non-zero terms of the Maclaurin series for $\mathrm{f}(x)$. [3]

(iii) Verify the correctness of the series found in part **(ii)** using the standard series from the List of Formulae (MF26). [2]

(iv) Use the series in part **(ii)** to approximate the value of $\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$.

Use your calculator to find an accurate value of $\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$. Why is the approximated value not very good? [3]

(v) Using the series obtained in part **(ii)**, deduce the Maclaurin series for $\cos[\ln(1+x)]$ in ascending powers of *x*, up to and including the term in x^2. [2]

[Solution]

(i) $\mathrm{f}(x)=\sin[\ln(1+x)]$ -----(1)

$$\mathrm{f}'(x)=\frac{1}{1+x}\cos[\ln(1+x)] \text{ -----(2)}$$

$$(1+x)\mathrm{f}'(x)=\cos[\ln(1+x)]$$

Differentiating wrt *x*:

$$(1+x)\mathrm{f}''(x)+\mathrm{f}'(x)=-\frac{1}{1+x}\sin[\ln(1+x)]$$

$$(1+x)^2\mathrm{f}''(x)+(1+x)\mathrm{f}'(x)+\mathrm{f}(x)=0 \quad \text{(shown) -----(3)}$$

"By further differentiation of the result in **(i)**" means you should use the result in the "same

(ii) Differentiating $(1+x)^2\mathrm{f}''(x)+(1+x)\mathrm{f}'(x)+\mathrm{f}(x)=0$ wrt *x*:

$$(1+x)^2\mathrm{f}'''(x)+2(1+x)\mathrm{f}''(x)+(1+x)\mathrm{f}''(x)+\mathrm{f}'(x)+\mathrm{f}'(x)=0$$

$$(1+x)^2\mathrm{f}'''(x)+3(1+x)\mathrm{f}''(x)+2\mathrm{f}'(x)=0 \text{ -----(4)}$$

When $x=0$,

from (1): $\mathrm{f}(0)=0$

from (2): $\mathrm{f}'(0)=1$

from (3): $\mathrm{f}''(0)+1+0=0\Rightarrow \mathrm{f}''(0)=-1$

from (4): $\mathrm{f}'''(0)+3(-1)+2(1)=0\Rightarrow \mathrm{f}'''(0)=1$

$$\mathrm{f}(x)=x(1)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(1)+\dots$$

$$=x-\frac{1}{2}x^2+\frac{1}{6}x^3+\dots$$

x is not small, no approximation should be made.

(iii) $\mathrm{f}(x)=\sin\left[\ln(1+x)\right]$

$$=\sin\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right)$$

$$=\left(x-\frac{x^2}{2}+\frac{x^3}{3}\right)-\frac{1}{6}\left(x-\frac{x^2}{2}+\frac{x^3}{3}\right)^3+\ldots$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{1}{6}x^3+\cdots$$

$$=x-\frac{x^2}{2}+\frac{1}{6}x^3+\cdots$$

Using **standard series**,

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$$

$$\sin x=x-\frac{x^3}{3!}+\ldots$$

x is not small, no approximation should be made.

$\therefore$ the series found in **(ii)** is correct till the terms in x^3

(iv) $\int_0^2 \mathrm{f}(x)\,\mathrm{d}x \approx \int_0^2 \left(x-\frac{1}{2}x^2+\frac{1}{6}x^3\right)\mathrm{d}x = 1.33$ (can use GC here)

Using GC, $\int_0^2 \mathrm{f}(x)\,\mathrm{d}x=\int_0^2 \sin\left(\ln(1+x)\right)\mathrm{d}x=1.153617\approx 1.15\ (3\text{ sf})$

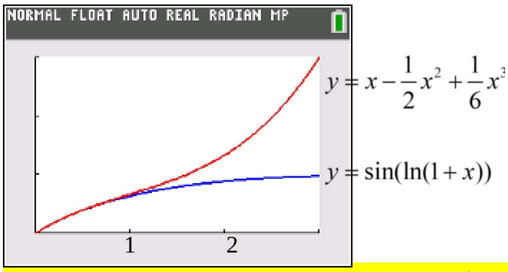
Percentage error $=\dfrac{1.33-1.153617}{1.153617}\times 100\% = 15.3\%$

As the percentage error is not small, hence the approximation is not so good.

OR

For $1\le x\le 2$, the **terms in higher powers of x** is not small and so are not negligible. Hence the approximation is not so good.

OR



For $1\le x\le 2$, the **graph** of $y=x-\frac{1}{2}x^2+\frac{1}{6}x^3$ **deviates** from that of $y=\sin(\ln(1+x))$.

Thus the **area under the graph** of $y=x-\frac{1}{2}x^2+\frac{1}{6}x^3$ **differs** from the area under the graph of $y=\sin(\ln(1+x))$. Hence the approximation is not so good.

(v) $\sin\left[\ln(1+x)\right]=x-\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots..$

Differentiate wrt x:

$$\frac{1}{1+x}\cos\left[\ln(1+x)\right]=1-x+\frac{1}{2}x^2+\ldots$$

$$\cos\left[\ln(1+x)\right]=(1+x)\left(1-x+\frac{1}{2}x^2+\ldots\right)$$

$$=1-\frac{1}{2}x^2+\ldots$$

x is not small, no approximation should be made.

4 **(a)** Relative to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors.

(i) It is given that $\mathbf{a}\times\mathbf{b}=\mathbf{0}$. Find $\mathbf{a}\,.\,\mathbf{b}$ in terms of $|\mathbf{b}|$. [2]

Draw a diagram to visualize

It is now given that $\mathbf{a}\times\mathbf{b}\neq\mathbf{0}$. Point P is the foot of the perpendicular from A to the line OB and the point Q is the foot of the perpendicular from B to the line OA. It is given that $AP = BQ$.

Length, not vector

(ii) Write down the lengths AP and BQ in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]
Hence show that $|\mathbf{a}|=|\mathbf{b}|$. [1]

(iii) Given also that the angle between $\mathbf{a}$ and $\mathbf{a}-\mathbf{b}$ is ϕ radians, show that $\mathbf{a}.\mathbf{b}=-|\mathbf{a}|^2\cos 2\phi$. [2]

(b) The pyramid $ORST$ has a triangular base ORS and height OT. The position vectors of R and S are $-4\mathbf{i}+\mathbf{j}+3\mathbf{k}$ and $5\mathbf{i}+\mathbf{j}$ respectively. Find the possible coordinates of T if the volume of the pyramid $ORST$ is 35 units^3. [5]

[*Volume of pyramid* $=\dfrac{1}{3}\times$ *base area* $\times$ *height*]

[Solution]

NOTE:
DO be very careful with the notation you are using:
Vector ($\underset{\sim}{a}$ or $\overrightarrow{OA}$) versus Magnitude of vector ($|\underset{\sim}{a}|$ or $|\overrightarrow{OA}|$ or OA)
Obviously, you cannot equate a vector to a magnitude (scalar).

(a) **(i)** $\underset{\sim}{a}\times\underset{\sim}{b}=\underset{\sim}{0}$

Note that cross product is a vector, ie $\underset{\sim}{0}$ zero

$\Rightarrow$ $\underset{\sim}{a}=k\,\underset{\sim}{b}$ where k is real (since $\underset{\sim}{a}\neq\underset{\sim}{0}$ and $\underset{\sim}{b}\neq\underset{\sim}{0}$)

$$\begin{aligned}\underset{\sim}{a}\,.\,\underset{\sim}{b}&=(k\,\underset{\sim}{b}).\underset{\sim}{b}\\&=k(\underset{\sim}{b}\,.\,\underset{\sim}{b})\\&=k(|\underset{\sim}{b}||\underset{\sim}{b}|\cos 0)\\&=k|\underset{\sim}{b}|^2\end{aligned}$$

For easier manipulation:
Substitute $\underset{\sim}{a}=k\,\underset{\sim}{b}$ first and factor out k.
Apply dot product formula only to $\underset{\sim}{b}\,.\,\underset{\sim}{b}$ to prevent complication (k can be +ve or –ve).

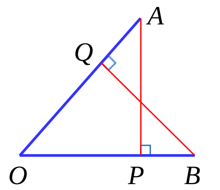
<u>**Alternative**</u> (Apply dot product formula to $k\underset{\sim}{b}\,.\,\underset{\sim}{b}$. Tedious, not encouraged)

$$\begin{aligned}\underset{\sim}{a}\,.\,\underset{\sim}{b}&=(k\,\underset{\sim}{b}).\underset{\sim}{b}\\&=|k\,\underset{\sim}{b}||\underset{\sim}{b}|\cos 0\ \ (\text{if } k>0)\quad\text{or}\quad |k\,\underset{\sim}{b}||\underset{\sim}{b}|\cos\pi\ \ (\text{if } k<0)\\&=k|\underset{\sim}{b}|^2\ \ (\text{if } k>0)\quad\text{or}\quad (-k)|\underset{\sim}{b}|^2(-1)\ \ (\text{if } k<0)\\&=k|\underset{\sim}{b}|^2\end{aligned}$$

Note $\underset{\sim}{a}=k\,\underset{\sim}{b}\Rightarrow \begin{array}{l}|\underset{\sim}{a}|=k\,|\underset{\sim}{b}|\ \text{ if } k>0\\ |\underset{\sim}{a}|=-k\,|\underset{\sim}{b}|\ \text{ if } k<0\end{array}$

(ii) $AP = \left|\underset{\sim}{a} \times \hat{\underset{\sim}{b}}\right| = \dfrac{\left|\underset{\sim}{a} \times \underset{\sim}{b}\right|}{\left|\underset{\sim}{b}\right|}$

$BQ = \left|\underset{\sim}{b} \times \hat{\underset{\sim}{a}}\right| = \dfrac{\left|\underset{\sim}{b} \times \underset{\sim}{a}\right|}{\left|\underset{\sim}{a}\right|}$



Given $AP = BQ$

$$\frac{\left|\underset{\sim}{a} \times \underset{\sim}{b}\right|}{\left|\underset{\sim}{b}\right|} = \frac{\left|\underset{\sim}{b} \times \underset{\sim}{a}\right|}{\left|\underset{\sim}{a}\right|}$$

$$\left|\underset{\sim}{a} \times \underset{\sim}{b}\right| \left|\underset{\sim}{a}\right| = \left|-(\underset{\sim}{a} \times \underset{\sim}{b})\right| \left|\underset{\sim}{b}\right|$$

$$\left|\underset{\sim}{a} \times \underset{\sim}{b}\right| \left|\underset{\sim}{a}\right| = \left|-1\right| \left|\underset{\sim}{a} \times \underset{\sim}{b}\right| \left|\underset{\sim}{b}\right|$$

$$\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right| \quad \text{(shown)}$$

While vector $\underset{\sim}{a} \times \underset{\sim}{b} = -(\underset{\sim}{b} \times \underset{\sim}{a})$, the magnitudes are equal: $\left|\underset{\sim}{a} \times \underset{\sim}{b}\right| = \left|\underset{\sim}{b} \times \underset{\sim}{a}\right|$.

Alternative

$$\frac{\left|\underset{\sim}{a} \times \underset{\sim}{b}\right|}{\left|\underset{\sim}{b}\right|} = \frac{\left|\underset{\sim}{b} \times \underset{\sim}{a}\right|}{\left|\underset{\sim}{a}\right|}$$

$$\frac{\left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\sin\theta}{\left|\underset{\sim}{b}\right|} = \frac{\left|\underset{\sim}{b}\right|\left|\underset{\sim}{a}\right|\sin\theta}{\left|\underset{\sim}{a}\right|}$$ where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$

$$\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right| \quad \text{(shown)}$$

(iii) Given angle between $\underset{\sim}{a}$ and $\underset{\sim}{a} - \underset{\sim}{b}$ is ϕ,

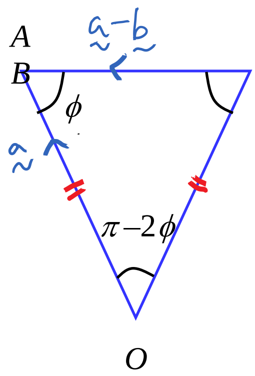
Note unit is radian and not

$\angle AOB = \pi - 2\phi$ (sum of angles in isosceles triangle)

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \left|\underset{\sim}{a}\right| \left|\underset{\sim}{b}\right| \cos(\pi - 2\phi)$

$= \left|\underset{\sim}{a}\right| \left|\underset{\sim}{a}\right| (-\cos 2\phi)$ since $\left|\underset{\sim}{b}\right| = \left|\underset{\sim}{a}\right|$

$= -\left|\underset{\sim}{a}\right|^2 \cos 2\phi$ (shown)



Alternative (long method, not encouraged)

$\underset{\sim}{a} \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = \left|\underset{\sim}{a}\right| \left|\underset{\sim}{a} - \underset{\sim}{b}\right| \cos(\phi)$

$\left|\underset{\sim}{a}\right|^2 - \underset{\sim}{a} \cdot \underset{\sim}{b} = \left|\underset{\sim}{a}\right| \left|\underset{\sim}{a} - \underset{\sim}{b}\right| \cos(\phi)$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \left|\underset{\sim}{a}\right|^2 - \left|\underset{\sim}{a}\right| \left|\underset{\sim}{a} - \underset{\sim}{b}\right| \cos(\phi)$ ----- (1)

Using cosine rule: (refer to diagram above)

$\left|\underset{\sim}{b}\right|^2 = \left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{a} - \underset{\sim}{b}\right|^2 - 2\left|\underset{\sim}{a}\right| \left|\underset{\sim}{a} - \underset{\sim}{b}\right| \cos(\phi)$

Since $\left|\underset{\sim}{b}\right| = \left|\underset{\sim}{a}\right|$,

$\left|\underset{\sim}{a} - \underset{\sim}{b}\right|^2 = 2\left|\underset{\sim}{a}\right| \left|\underset{\sim}{a} - \underset{\sim}{b}\right| \cos(\phi)$

$\left|\underset{\sim}{a} - \underset{\sim}{b}\right| = 2\left|\underset{\sim}{a}\right| \cos(\phi)$ ----- (2)

Sub (2) into (1),

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \left|\underset{\sim}{a}\right|^2 - \left|\underset{\sim}{a}\right| (2\left|\underset{\sim}{a}\right| \cos(\phi)) \cos(\phi)$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \left|\underset{\sim}{a}\right|^2 (1 - 2\cos^2(\phi))$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = -\left|\underset{\sim}{a}\right|^2 \cos(2\phi)$ (shown)

(b) Volume $= \frac{1}{3}(\text{Area } \Delta ORS)(OT) = 35$

$$\frac{1}{3}\left(\frac{1}{2}\left|\overrightarrow{OR}\times\overrightarrow{OS}\right|\right)(OT) = 35$$

$$\frac{1}{6}\left|\begin{pmatrix}-4\\1\\3\end{pmatrix}\times\begin{pmatrix}5\\1\\0\end{pmatrix}\right|(OT) = 35$$

$$\frac{1}{6}\left|-3\begin{pmatrix}1\\-5\\3\end{pmatrix}\right|(OT) = 35$$

$$OT = \frac{2(35)}{\sqrt{35}} = 2\sqrt{35}$$

Normal to ORS, $\underset{\sim}{n} = \overrightarrow{OR}\times\overrightarrow{OS} = -3\begin{pmatrix}1\\-5\\3\end{pmatrix}$. Therefore $\overrightarrow{OT} = k\begin{pmatrix}1\\-5\\3\end{pmatrix}$

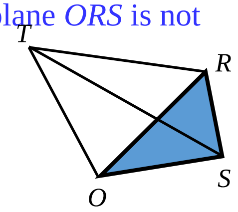
$$\left|\overrightarrow{OT}\right| = \left|k\begin{pmatrix}1\\-5\\3\end{pmatrix}\right| = 2\sqrt{35}$$

$$|k|\sqrt{35} = 2\sqrt{35} \quad \Rightarrow \quad k = \pm 2$$

$$\overrightarrow{OT} = \pm 2\begin{pmatrix}1\\-5\\3\end{pmatrix} = \pm\begin{pmatrix}2\\-10\\6\end{pmatrix}$$

Note that point T is NOT 'vertically' above the origin O as the plane ORS is not horizontal.

Hence $\overrightarrow{OT} \neq \begin{pmatrix}0\\0\\k\end{pmatrix}$



Coordinates of T are $(2, -10, 6)$ or $(-2, 10, -6)$

Alternative

$$\overrightarrow{OT} = \pm 2\sqrt{35}\frac{\mathbf{n}}{|\mathbf{n}|} \quad = \pm 2\sqrt{35}\frac{\begin{pmatrix}1\\-5\\3\end{pmatrix}}{\sqrt{35}}$$

$$= \begin{pmatrix}2\\-10\\6\end{pmatrix} \quad \text{or} \quad -\begin{pmatrix}2\\-10\\6\end{pmatrix}$$

Coordinates of T are $(2, -10, 6)$ or $(-2, 10, -6)$

5 In a chicken farm, eggs are packed into boxes of 12. In the transportation of the eggs from the farm to a supermarket, it is found that on average, k eggs out of 10 eggs are broken on arrival at the supermarket. A box of eggs is randomly chosen for inspection. Find, in terms of k,

(i) the expected number of eggs that are broken in the box, [1]

(ii) the probability that the 8th egg inspected is the 2nd egg that is broken. [2]

It is given that $k = 0.8$.

(iii) Find the probability that there are not more than 2 broken eggs in the box. [2]

(iv) If n boxes are randomly selected for inspection, find the least value of n such that the probability that at most 3 boxes will have at least 3 broken eggs is less than 0.9. [3]

[Solution]

(i) P(an egg is broken) $= \frac{k}{10}$

np

Expected number of broken eggs in a box $= 12 \times \frac{k}{10} = \frac{6}{5}k$

(ii) Required probability $= \frac{k}{10} \times \left(1 - \frac{k}{10}\right)^6 \times {}^7C_1 \times \frac{k}{10} = \frac{7}{100}k^2\left(1 - \frac{k}{10}\right)^6$

$\frac{k}{10} \times \left(1 - \frac{k}{10}\right)^6$: P(first egg is broken and the next 6 eggs are not broken)

${}^7C_1$: The 1st broken egg can be any of the first 7 eggs

$\frac{k}{10}$: P(8th egg is broken)

(iii) Let X be the number of eggs (out of 12) that are broken.
$X \sim B(12, 0.08)$
$P(X \le 2) = 0.934$

(iv) Let Y be the number of boxes (out of n) with at least 3 broken eggs.
$P(X \ge 3) = 1 - P(X \le 2) = 0.065195$

$Y \sim B(n, 0.065195)$

5 sf for intermediate

$P(Y \le 3) < 0.9$

Using GC,

n	$P(Y \le 3)$
27	0.90426
28	0.89378

Least value of n is 28

6 Javier and Kelvin played a game with a pack of ten cards numbered 1, 2, 3, …, 10. They take turns at drawing two cards each from the pack at random, one card at a time without replacement, and the two cards drawn are replaced before the next player draws. The player who first gets two cards whose sum is 10 wins the game. If he does not win, he replaces the two cards and the other player then takes his turn. The game carries on until a winner is decided.

On their first game, Javier gets to draw first.

(i) Show that the probability that Javier wins the game on his first draw is $\frac{4}{45}$. [2]

(ii) Find the probability that Kelvin wins the game. [3]

On their second game, they toss a biased coin to decide on the player to draw first. Using this biased coin, Kelvin is twice as likely as Javier to draw first.

(iii) Find the probability that Javier wins the game. [3]

[Solution]

(i) P(Javier wins the game on his first draw | Javier draws first)

$= \mathrm{P}(\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}) \times 2$

$$= \left(\frac{1}{10} \times \frac{1}{9}\right) \times 8 = \frac{4}{45}$$

(ii) P(Kelvin wins the game | Javier draws first)

$= \mathrm{P}(\mathrm{J}'\,\mathbf{K}) + \mathrm{P}(\mathrm{J}'\,\mathrm{K}'\,\mathrm{J}'\,\mathbf{K}) + \mathrm{P}(\mathrm{J}'\,\mathrm{K}'\,\mathrm{J}'\,\mathrm{K}'\,\mathrm{J}'\,\mathbf{K}) + \ldots.$

Infinite

$$= \frac{41}{45} \times \frac{4}{45} + \left(\frac{41}{45}\right)^3 \times \frac{4}{45} + \left(\frac{41}{45}\right)^5 \times \frac{4}{45} + \ldots$$

$$= \frac{41}{45} \times \frac{4}{45}\left(1 + \left(\frac{41}{45}\right)^2 + \left(\frac{41}{45}\right)^4 + \ldots\right)$$

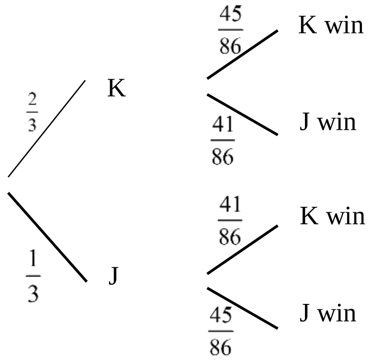
Sum to infinity of a G.P.

$$= \frac{\frac{41}{45} \times \frac{4}{45}}{1 - \left(\frac{41}{45}\right)^2}$$

$$= \frac{41}{86} \quad \text{or} \quad 0.477 \text{ (3 s.f.)}$$

Make use of (ii) to do

(iii) From **(ii)**, $\mathrm{P}(Y \text{ wins} \mid X \text{ draws first}) = \frac{41}{86}$



$$= \frac{1}{3}\mathrm{P}(\text{J wins} \mid \text{J draws first}) + \frac{2}{3}\mathrm{P}(\text{J wins} \mid \text{K draws first})$$

$$= \frac{1}{3}\left(1 - \frac{41}{86}\right) + \frac{2}{3}\left(\frac{41}{86}\right)$$

$$= \frac{127}{258}$$

7 The diagram shows a 3×3 grid of 9 cells, numbered 1 to 9.

Letters (and numerals) are always assumed to be identical unless otherwise

1	2	3
4	5	6
7	8	9

Having numbered cells indicates that we do not consider rotational symmetry. However it is not exactly the same as "numbered chairs" in a circular arrangement

The nine letters T, T, T, T, J, C, C, C, C are to be arranged in the grid, with each cell occupied by one letter. Find the number of ways to arrange the nine letters if

(i) there are no other restrictions, [2]

(ii) the cells at the four corners of the grid must each be occupied by a 'T'. [1]

Two letters are considered adjacent to each other if they occupy cells that are above, below, to the left or to the right of each other. Find the number of ways to arrange the nine letters if

(iii) all 'T's must not be adjacent to another 'T' and all 'C's must not be adjacent to another 'C', [2]

(iv) at least two 'T's must be adjacent and at least two 'C's must be adjacent. [3]

(${}^5C_1$: Choose cell for 'J')

[Solution]

(i) No. of ways = $\dfrac{9!}{4!4!} = 630$

Some students multiply this by 9, probably because the cells are numbered 1 to 9.
Do not solve problems on autopilot.
Think about the scenario presented.

(ii) No. of ways = ${}^5C_1 = 5$ or $\dfrac{5!}{4!} = 5$

T	2	T
4	5	6
T	8	T

${}^5C_1$: Select from cells 2, 4, 5, 6 or 8 for

(iii) No. of ways = ${}^2C_1 \times {}^5C_1 = 10$
(${}^2C_1$: Cells 2, 4, 6 and 8 all occupied by either 'T' or 'C'; ${}^5C_1$: choose cell for 'J')

1	T	3
T	5	T
7	T	9

${}^2C_1$: Select either "T" or "C" to occupy cells 2, 4, 6 and 8.
${}^5C_1$: Select from cells 1, 3, 5, 7, or 9 for "J"

(iv) Define events A: no two 'T's are adjacent
and B: no two 'C's are adjacent.

To find n(A):

1	T	3
T	5	T
7	T	9

Case 1: 'T's occupy cells 2, 4, 6 and 8
Number of ways no two 'T's are adjacent
= number of ways to select from cells 1, 3, 5, 7 or 9 for "J"
= 5

(T)	2	(T)
4	(T)	6
(T)	8	(T)

Case 2: ‘T’s occupy 4 cells from among 1, 3, 5, 7 and 9
Number of ways no two ‘T’s are adjacent

$= {}^5C_4 \times {}^5C_1 = 25$

Choose 4 of 1, 3, 5, 7 or 9 — Choose cell for ‘J’ from 5 remaining cells

$n(A) = (1 \times {}^5C_1) + ({}^5C_4 \times {}^5C_1) = 30$

By symmetry, $n(B) = 30$

Then $n(A \cap B)$ = answer from (iii) = 10.

Number of ways with at least 2 ‘T’s adjacent and at least 2 ‘C’s adjacent
= Total number without restriction − $n(A \cup B)$
$= 630 - (n(A) + n(B) - n(A \cap B))$
$= 630 - (30 + 30 - 10) = 580$

9 In a TV sports entertainment competition, Spartan Warrior, each contestant has to complete two obstacle stations X and Y in the shortest time possible. It has been found that the time taken to complete station X is normally distributed with mean μ minutes and variance 1.29 minutes^2, and the time taken to complete station Y is also normally distributed with mean 6.5 minutes and variance 1.06 minutes^2. Assume that each contestant begins station Y immediately after completing station X.

(i) If the probability that a randomly chosen contestant takes more than 22.5 minutes to complete both stations is less than 0.0102, find the range of values that μ can take. State an assumption you made in your calculations. [4]

For the remainder of the question, assume that $\mu = 12.3$.

(ii) Find the probability that the time taken by a randomly chosen contestant to complete station X is greater than twice the time taken to complete station Y. [3]

Assume that a contestant begins immediately after the previous contestant completes both stations.

(iii) Find the maximum number of contestants that can be invited to a recording session if the probability that the total time taken for all contestants to complete both stations is within 7 hours is more than 0.99. [3]

(iv) During the broadcast of each episode, the sports channel will broadcast 3 randomly chosen contestants completing both stations without editing. Find the probability that, in one episode, the time taken by one of the 3 randomly chosen contestants to complete both stations is less than 17 minutes and the time taken by the remaining 2 contestants to complete both stations is more than 20 minutes each. [2]

Solution

Let A and B be the time taken to complete station X and Y respectively

$A \sim \text{N}(\mu, 1.29)$ and $B \sim \text{N}(6.5, 1.06)$

(i) Let $T = A + B \sim \text{N}(6.5 + \mu, 2.35)$

Should not use GC "Table" method by trying a few values of μ - Use "Table" method only when the unknown is and integer value

$\text{P}(A + B > 22.5) < 0.0102$

$$\text{P}\left(Z > \frac{16-\mu}{\sqrt{2.35}}\right) < 0.0102$$

$$\frac{16-\mu}{\sqrt{2.35}} > 2.318908468$$

InvNorm(1 – 0.0102) = 2.318908468
Draw a standard normal curve to visualise and understand how this inequality is established – you must get this correct !

$\mu < 12.44518063$

i.e. $\mu < 12.4$ (correct to 3 sf)

<u>**Assumption:**</u> The times taken (for each contestant) to complete station X and Y are independent.

(ii) Note $A \sim \text{N}(12.3, 1.29)$

$A - 2B \sim \text{N}(12.3 - 2\times 6.5,\ 1.29 + 2^2\times 1.06)$

i.e. $A - 2B \sim \text{N}(-0.7, 5.53)$

$\text{P}(A - 2B > 0) = 0.415811\ 0.3829775 \approx 0.383$ (3sf)

(iii) Note $T \sim N(18.8, 2.35)$
$T_1 + T_2 + \ldots + T_n \sim N(18.8n, 2.35n)$
$P(T_1 + T_2 + \ldots + T_n < 420) > 0.99$

Total time taken for n contestants is not nT .

From GC,

n	$P(T_1 + T_2 + \ldots + T_n < 420)$
21	$0.9998 > 0.99$
22	$0.8133 < 0.99$

If you use GC, you must show this table as your "method"

Therefore maximum number of contestants invited for recording is 21.

Alternative method:

$P(T_1 + T_2 + \ldots + T_n < 420) > 0.99 \quad \Rightarrow \quad P\left(Z < \frac{420 - 18.8n}{\sqrt{2.35n}}\right) > 0.99$

$\frac{420 - 18.8n}{\sqrt{2.35n}} > 2.326347$

Use GC, max integer $n = 21$

(iv) Required probability $= P(T < 17)\, P(T > 20)^2 \times \frac{3!}{2!}$

$= 0.0170$ (3 sf)

The event "T < 17" can happen amongst one of the 3

10 Anand and Charlie decide to do some exercise and have fun at the same time.

Before the start of each set of exercise, Anand will place 4 blue balls and 5 green balls, identical apart from their colour, into a bag. Charlie will then randomly select 4 balls from the bag without replacement.

X denotes the number of blue balls chosen by Charlie in one set of exercise.

For each blue ball selected, Anand will perform 5 abdominal crunches.
For each green ball selected, Charlie will perform 3 chin-ups.

(i) Show that $\mathrm{P}(X = 3) = \dfrac{10}{63}$ and tabulate the probability distribution of X. [3]

C denotes the number of chin-ups performed by Charlie in 1 set of exercise.

(ii) Without using a graphing calculator, show that $\mathrm{Var}(X) = \dfrac{50}{81}$ and, hence or otherwise, find $\mathrm{Var}(C)$. [5]

(iii) Anand and Charlie decide to do 3 sets of exercise per day for a period of 30 days. Find the approximate probability that during this period, the total number of abdominal crunches performed by Anand is at least 240 greater than the total number of chin-ups performed by Charlie. [4]

[Solution]

(i) $\mathrm{P}(X = 3) = \dfrac{{}^4C_3\,{}^5C_1}{{}^9C_4} = \dfrac{20}{126} = \dfrac{10}{63}$ (shown) or $\dfrac{4}{9}\times\dfrac{3}{8}\times\dfrac{2}{7}\times\dfrac{5}{6}\times 4$

x	0	1	2	3	4
$\mathrm{P}(X = x)$	$\dfrac{{}^5C_4}{{}^9C_4} = \dfrac{5}{126}$	$\dfrac{{}^4C_1\,{}^5C_3}{{}^9C_4} = \dfrac{20}{63}$	$\dfrac{{}^4C_2\,{}^5C_2}{{}^9C_4} = \dfrac{10}{21}$	$\dfrac{10}{63}$	$\dfrac{{}^4C_4}{{}^9C_4} = \dfrac{1}{126}$

Do Not miss out the case when $X = 0$ in the probability distribution of X.
Checking : Add up all probabilities must be equal to 1

(ii) $\mathrm{E}(X) = 0\left(\dfrac{5}{126}\right)+1\left(\dfrac{20}{63}\right)+2\left(\dfrac{10}{21}\right)+3\left(\dfrac{10}{63}\right)+4\left(\dfrac{1}{126}\right)=\dfrac{16}{9}$

This is "show .." Must show working of how to obtain $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$

$\mathrm{E}(X^2)$

$= 0^2\left(\dfrac{5}{126}\right)+1^2\left(\dfrac{20}{63}\right)+2^2\left(\dfrac{10}{21}\right)+3^2\left(\dfrac{10}{63}\right)+4^2\left(\dfrac{1}{126}\right)=\dfrac{34}{9}$

$\mathrm{Var}(X) = \mathrm{E}(X^2)-[\mathrm{E}(X)]^2 = \dfrac{34}{9}-\left(\dfrac{16}{9}\right)^2 = \dfrac{50}{81}$

"Hence" Method:

Observe that $C = 3(4 - X) = 12 - 3X$

$\mathrm{Var}(C) = \mathrm{Var}(12 - 3X) = 9\mathrm{Var}(X) = \dfrac{50}{9}$

Total number of balls selected = **4 Not 9**
No. of green balls = $4-X$
Note: $C \neq 3X$, $C \neq 3(9-X)$

"Otherwise" Method 1 (Tedious!):

Use probability distribution of C to find $\mathrm{Var}(C) = \mathrm{E}(C^2)-[\mathrm{E}(C)]^2$

C	0	3	6	9	12
$\mathrm{P}(C = c)$	$\dfrac{1}{126}$	$\dfrac{10}{63}$	$\dfrac{10}{21}$	$\dfrac{20}{63}$	$\dfrac{5}{126}$

"Otherwise" Method 2 (Tedious)

Use probability distribution of G (where G = No. of green balls selected) to find $\text{Var}(G) = \text{E}\left(G^2\right) - \left[\text{E}\left(G\right)\right]^2$ then use $C = 3G$ to find Var(C).

g	0	1	2	3	4
$\text{P}(G = g)$	$\frac{1}{126}$	$\frac{10}{63}$	$\frac{10}{21}$	$\frac{20}{63}$	$\frac{5}{126}$

(iii) Let $W = X_1 + X_2 + \ldots + X_{90}$

Consider 3×30 exercises and Not 30 days

Since $\boldsymbol{n}$ **= 90** is large, $W \sim \text{N}\left(90\left(\frac{16}{9}\right), 90\left(\frac{50}{81}\right)\right)$ approximately **by Central Limit Theorem.** i.e. $W \sim \text{N}\left(160, \frac{500}{9}\right)$ approximately

Note:
S and T are NOT independent!!
$S - T$ does not follows normal distribution.

Let A and C be number of crunches and chin ups done by Anand and Charlie respectively.

Let $S = A_1 + A_2 + \ldots + A_{90} = 5W$

Let $T = C_1 + C_2 + \ldots + C_{90}$

$= 3((4 - X_1) + (4 - X_2) + \ldots + (4 - X_{90})) = 1080 - 3W$

$\text{P}(S - T \geq 240)$

$= \text{P}(5W - (1080 - 3W) \geq 240)$

$= \text{P}(W \geq 165)$

$= 0.251$ (3sf)

This is about selection of sample and not about percentage weight loss of steak after cooking

11 It is found that certain kinds of meat lose weight as a result of being cooked. A restaurant chef is prepared to accept up to 10% loss but suspects that the recent consignments have a higher percentage weight loss. She decides to carry out a hypothesis test on a random sample of steaks.

(i) Explain the meaning of 'a random sample' in the context of the question and why the chef should sample a large number of steaks. [2]

(ii) State suitable hypotheses for the test, defining any symbols that you use. [2]

The chef takes a random sample of 40 steaks, and calculate the percentage weight loss of each steak, x (in percent). The mean and variance of the percentage weight loss of the 40 steaks are 10.48% and 3.37% respectively.

(iii) Test, at the 5% significance level, to determine whether the sample supports the chef's suspicion. [3]

The chef carries out another test using 100 readings obtained from chefs of other restaurants. The percentage weight loss of each steak, y (in percent), is summarised by:

$$\sum y = 1055.2,\ \sum y^2 = k\ .$$

(iv) Find the range of values of k such that the chef's suspicion is not valid at the 5% level of significance, giving your answer correct to 2 decimal places. [5]

Solution

(i) A random sample means that each of the steaks is **selected independently** from each other *and* **each** steak has an **equal chance** of being **selected**.

Since the distribution of the percentage weight loss of the steak after cooking (population) is not known, the chef should take a sample of at least 30 steaks so that the distribution of the sample mean of percentage weight loss of steak after cooking can be approximated by a normal distribution using the Central Limit Theorem.

(ii) Let X(in percent) be the percentage weight loss as a result of cooking and $\mu\%$ be the population mean percentage weight loss of steaks as a result of being cooked.

$H_0 : \mu = 10$
$H_1 : \mu > 10$

Must clearly written. Not mean and not weight of steak

Must indicate clearly

Must define correctly and tally with the figure.

Central Limit Theorem does not approximate $X \sim \mathrm{N}(10, 3.4561)$

Not in critical region means $z_{cal} < z_{critical}$.
Incorrect: $z_{cal} \le z_{critical}$ or $z_{cal} > z_{critical}$

... the ... defined in the question. Hence not 0.1 or 10%. This is also not sample mean also (10.48).

(iii) Level of significance: 5%

Unbiased estimate of population variance = $\frac{40}{39}$(

Under H_0, $\bar{X} \sim \mathrm{N}\left(10, \frac{3.45641}{40}\right)$ approximately by Central Limit Theorem as $n = 40$ is large.

Cannot use sample variance.

Test Statistic. $Z = \frac{\bar{X} - 10}{\sqrt{3.45641/40}} \sim \mathrm{N}(0,1)$ approximately

Many did not include sample size

From GC, p-value = 0.051235 > 0.05

Since p-value > significance value, we do not reject H_0
There is <u>insufficient evidence</u> at 5% level of significance to conclude that the sample confirms the chief's suspicion.

(iv) Let Y (in percent) be the percentage weight loss as a result of cooking
$H_0 : \mu = 10$
$H_1 : \mu > 10$
Level of significance: 5%

Under H_0, $\bar{Y} \sim \mathrm{N}\left(10, \frac{s^2}{100}\right)$ approximately by Central Limit Theorem as $n = 100$ is large.

Test Statistic. $Z = \frac{\bar{Y} - 10}{s/\sqrt{100}} \sim \mathrm{N}(0,1)$ approximately

$$\bar{y} = \frac{1055.2}{100} = 10.552$$

Do not reject H_0 at 5% level of significance:

$$z_{cal} = \frac{10.552 - 10}{s/\sqrt{100}} < 1.6448536$$

Not in critical region means $z_{cal} < z_{critical}$.
Incorrect: $z_{cal} \le z_{critical}$ or $z_{cal} > z_{critical}$

Critical region always depends on H_1.

$$s > 3.355921767$$

$$\frac{1}{99}\left(k - \frac{(1055.2)^2}{100}\right) > 3.355921767^2$$

$$k > 12249.43$$