2020 JC1 H2 Maths Promotional Exam:

1 Amy, Bobby and Carla went shopping for cupcakes. Amy paid \$5 for each of her cupcakes, Bobby paid \$4 for each of his cupcakes, and Carla paid \$3 for each of hers. As a result, although Amy bought 4 cupcakes fewer than Bobby and Carla combined, she paid \$14 more than Bobby and Carla combined.

If, instead, Amy and Bobby had paid \$3 for each of their cupcakes, they would have paid \$46 less in total. Find the total number of cupcakes bought by Amy, Bobby and Carla. [4]

2 **(i)** Solve exactly the inequality $\dfrac{x^2-x-1}{x+1} \le 1$. [3]

(ii) Hence, solve exactly the inequality $\dfrac{x^2+x-1}{1-x} \le 1$. [2]

3 **(i)** Sketch the curve with equation

$$y = \frac{x^2+9x-5}{x+10},$$

indicating clearly the equations of any asymptotes, the coordinates of any points where the curve crosses the axes and of any turning points. [4]

(ii) Hence, find the range of values of k, where k is real, such that the equation

$$\frac{x^2+9x-5}{x+10} = kx+10k-11$$

has no solution. Show your reasoning clearly. [2]

4 Without using a calculator, show that the curve defined by the parametric equations

$$x = t^2+t, \qquad y = t^3+t^2$$

intersects the line $y = x$ at two distinct points. Prove that this line is the normal to the curve at exactly one of these points. [6]

5 **(a)** The variable vector $\mathbf{u}$ satisfies the following equations:

$$\mathbf{u} \cdot \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = -6 \text{ and}$$

$$\mathbf{u} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = k\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \text{ for } k \in \mathbb{R}, k \neq 0.$$

(i) Explain why $\mathbf{u}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}=0$. [1]

(ii) Hence or otherwise, find the set of vectors **u** and describe this set geometrically. [3]

(b) The points A, B and C have distinct non-zero position vectors **a**, **b** and **c** respectively and the vectors satisfy the equation $\mathbf{c}=\lambda\mathbf{a}+(1-\lambda)\mathbf{b}$ where $\lambda\in\mathbb{R}, \lambda\neq 0, \lambda\neq 1$. Prove that the points A, B and C are collinear. [2]

(c) Given that the point P has a non-zero position vector **p** and that the plane Π has equation $\mathbf{r}\cdot\mathbf{n}=0$, where **n** is a unit vector, state the geometrical meaning of $|\mathbf{p}\cdot\mathbf{n}|$ in relation to the point P and the plane Π. [1]

6 **(a)** Find $\displaystyle\int\frac{-5-2x}{\sqrt{7-6x-x^2}}\,\mathrm{d}x$. [4]

(b) Find $\int 2\sin kx\sin x\,\mathrm{d}x$, where $k\in\mathbb{Z}, k\geq 2$. [3]

7 The curve C has equation

$$\tan^{-1}y=x\ln x^2-2x-y\text{, where } x\in\mathbb{R}, x>0.$$

(i) Show that $\left(2+y^2\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}=2(1+y^2)\ln x.$ [2]

(ii) Hence find the coordinates of the stationary point of C. [3]

(iii) By differentiating the result shown in part **(i)**, determine the nature of the stationary point. [3]

(iv) State the equation of the normal to the curve C that is parallel to the y-axis. [1]

8 **(i)** Using the identity $4\cos^3\theta=3\cos\theta+\cos 3\theta$, show that

$$\frac{4\cos^3\left(3^r x\right)}{(-3)^r}=3\left[\frac{\cos\left(3^r x\right)}{(-3)^r}-\frac{\cos\left(3^{r+1}x\right)}{(-3)^{r+1}}\right].$$ [2]

(ii) Hence, show that $\displaystyle\sum_{r=1}^{n}\frac{\cos^3\left(3^r x\right)}{(-3)^r}=-\frac{1}{4}\cos 3x-\frac{3}{4}\left[\frac{\cos\left(3^{n+1}x\right)}{(-3)^{n+1}}\right]$. [3]

(iii) The Squeeze Theorem states that if $\dfrac{p}{\mathrm{f}(n)} \le \dfrac{\cos\left(3^{n+1}x\right)}{(-3)^{n+1}} \le \dfrac{q}{\mathrm{f}(n)}$ for all positive integers n, and $\lim\limits_{n\to\infty}\dfrac{p}{\mathrm{f}(n)} = \lim\limits_{n\to\infty}\dfrac{q}{\mathrm{f}(n)} = 0$, then $\lim\limits_{n\to\infty}\dfrac{\cos\left(3^{n+1}x\right)}{(-3)^{n+1}} = 0$ too. By considering the minimum and maximum values of the cosine function, use the Squeeze Theorem to explain why $\sum\limits_{r=1}^{\infty}\dfrac{\cos^3\left(3^r x\right)}{(-3)^r} = -\dfrac{1}{4}\cos 3x$. [3]

(iv) Evaluate $\sum\limits_{r=1}^{\infty}\dfrac{\cos^3\left(3^{r-1}\pi\right)}{(-3)^r}$. [1]

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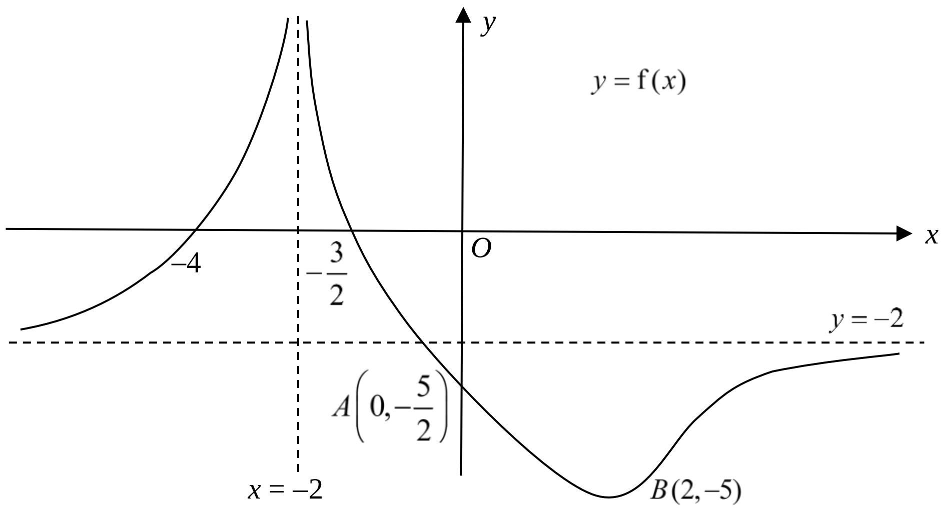

The diagram above shows the curve of $y = \mathrm{f}(x)$ with horizontal asymptote $y = -2$. The curve crosses the x-axis at the points $(-4, 0)$ and $\left(-\dfrac{3}{2}, 0\right)$, and crosses the y-axis at point A with coordinates $\left(0, -\dfrac{5}{2}\right)$. The curve has a turning point at B with coordinates $(2, -5)$. It is given that $\mathrm{f}'(0) = -\dfrac{4}{3}$.

(i) On separate diagrams, showing in each case where appropriate, the coordinates of the points corresponding to A and B, the x-intercepts, and the equations of the asymptotes, sketch the graphs of

(a) $y = \mathrm{f}'(x)$, [3]

(b) $y = \dfrac{1}{\mathrm{f}(x)}$. [4]

(ii) By adding a suitable graph to your diagram in part **(i)(b)**, or otherwise, solve the inequality

$$\frac{10}{\mathrm{f}(x)} + x + 4 \le 0.$$ [3]

10 Functions f and g are defined by

$$\mathrm{f}: x \mapsto x^3 - 7x^2 - 5x + 11, \quad x \in \mathbb{R}, x \geq k,$$

$$\mathrm{g}: x \mapsto (x+1)^2 + 2, \quad x \in \mathbb{R}.$$

(i) Let $k = 1$.

(a) Show that f does not have an inverse. [2]

(b) Determine whether the composite function fg exists. [2]

(ii) Find the value of *k* given that f^{-1} exists and that the domain of f^{-1} is $x \in \mathbb{R}, x \geq -24$. [2]

(iii) Let $k = 6$.

(a) Show algebraically that $\mathrm{f}'(x) > 0$ for all values of *x* in the domain of f. [2]

(b) Solve the equation $\mathrm{gf}^{-1}(x) = 83$. [3]

11 **(i)** On 1 January 2020, Ms Eu took a study loan of \$20,000 from BOPS bank to fund her part-time studies. Based on the loan agreement, Ms Eu will make a yearly repayment of \$*x* on 30 December of each year, starting from 30 December 2020. Furthermore, an interest rate of 4% per year is applied on the amount outstanding as at 31 December of each year, starting from 31 December 2020.

(a) Show that the amount outstanding at the end of the n^{th} year is

$$1.04^n(20000) - kx\left(1.04^n - 1\right),$$

where *k* is a constant to be determined. [3]

(b) Determine the value of *x* if Ms Eu intends to repay the loan fully at the end of 2024. [3]

(ii) While embarking on her part-time studies, Ms Eu started a new job and received her first salary of \$3,300 on 1 July 2020. She saved \$1,600 from her first pay and deposited it into an account that paid no interest. Ms Eu intends to increase the amount saved from her salary by \$50 per month, until her account balance exceeds \$35,000. She will then transfer the full amount in this account into a fixed deposit in BOPS Bank immediately. She will keep the money for 12 months in the fixed deposit which earns a simple interest of 1.5% per year.

(a) Find the month and year in which Ms Eu will withdraw the fixed deposit. [3]

(b) Determine the value of the fixed deposit that Ms Eu will withdraw. [1]

(iii) After withdrawing the fixed deposit, Ms Eu has two options:

Option 1: Deposit \$30,000 in a fixed deposit that earns a simple interest of 1.5% per year.

Option 2: Invest \$30,000 in a fund that will yield a compound interest of 1.2% per year.

Showing your working clearly, explain which option would be more suitable for Ms Eu if she wants to grow the value of this \$30,000 in the long term. [2]

12 Stereophotogrammetry is a method of determining coordinates of points in the three-dimensional (3-D) replication of physical scenes. It relies on using multiple images taken by digital cameras from different positions.

In a simplistic model for this process, the camera sensors are represented by planes with finite size. A *ray of sight* for a particular point in the physical scene is defined as the line passing through its image point on the camera sensor and the focal point of the camera. The 3-D coordinates of a particular point is the intersection of the rays of sight from the different cameras.

For a particular set up, Camera 1 has focal point, F_1 at $(20, 5, 10)$ and Camera 2 has focal point, F_2 at $(-20, 5, 10)$. The image point of a point *A*, the highest tip of a flag pole, on Camera 1 is $A_1\left(\frac{61}{3}, 0, \frac{28}{3}\right)$, and the image point of *A* on Camera 2 is $A_2\left(-\frac{67}{3}, 0, \frac{28}{3}\right)$.

(i) Find the vector equations of l_1 and l_2, the rays of sight for point *A* from the Cameras 1 and 2 respectively, and hence find the coordinates of point *A*. [4]

The base of the flag pole is known to be on the plane *P* that contains the point $D(75, 90, -50)$ and the line *L* with equation $\mathbf{r} = \begin{pmatrix} 23 \\ 90 \\ -46 \end{pmatrix} + s\begin{pmatrix} -1 \\ 6 \\ 1 \end{pmatrix}, s \in \mathbb{R}$.

(ii) Show that the vector equation of the plane *P* is $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 13 \end{pmatrix} = -755$. [3]

The flag pole was erected perpendicular to the base plane *P*.

(iii) Find the coordinates of the point *B*, the base of the flag pole. [3]

(iv) Hence or otherwise, find the length of the flag pole. [2]

(v) Given that the horizontal plane in this model is the *x*-*y* plane, find the angle of incline for the plane *P* from the horizontal plane. [2]