A-LEVEL H2 MATH 2020 – PAPER 2

Section A

Question 1

[Ans: $y=\frac{5}{2}x^2-5x+\frac{1}{2}$]

Let the equation of the quadratic curve be $y=ax^2+bx+c$.

$$\frac{dy}{dx}=2ax+b$$

At $(1,-2)$,

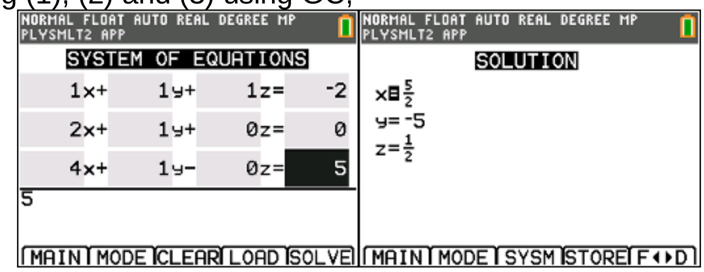
$$a(1)^2+b(1)+c=-2$$
$$a+b+c=-2 \quad (1)$$

$$\frac{dy}{dx}=0$$
$$2a(1)+b=0$$
$$2a+b=0 \quad (2)$$

When $x=2$,

$$\frac{dy}{dx}=5$$
$$2a(2)+b=5$$
$$4a+b=5 \quad (3)$$

Solving (1), (2) and (3) using GC,



$$a=\frac{5}{2},\ b=-5,\ c=\frac{1}{2}$$

$\therefore$ quadratic equation: $y=\frac{5}{2}x^2-5x+\frac{1}{2}$

Question 2

[Ans: (a)(i)(A) increasing (B) constant, 5 (ii) $p=11$ (b)(i) $b=7$ (ii) $v_5=5a+28$
(c)(i) $3n^2-25n+16$ (ii) $m=10$]

(a) (i) When $p=7$,

$u_1=7$

$u_2=2u_1-5=2(7)-5=9$

$u_3=2u_2-5=2(9)-5=13$

$u_4=2u_3-5=2(13)-5=21$

$\therefore$ from observation, the sequence is increasing.

When $p=5$,

$u_1=5$

$u_2=2u_1-5=2(5)-5=5$

$u_3=2u_2-5=2(5)-5=5$

$u_4=2u_3-5=2(5)-5=5$

$\therefore$ from observation, every term in the sequence is 5.

(ii) $u_1=p$

$u_2=2u_1-5=2p-5$

$u_3=2u_2-5=2(2p-5)-5=4p-15$

$u_4=2u_3-5=2(4p-15)-5=8p-35$

$u_5=2u_4-5=2(8p-35)-5=16p-75$

$\therefore 16p-75=101 \Rightarrow p=11$

(b) (i) $v_3=v_1+2v_2-7=a+2b-7$

$v_4=v_2+2v_3-7=b+2(a+2b-7)-7=2a+5b-21$

$v_4=2v_3$

$2a+5b-21=2(a+2b-7)$

$2a+5b-21=2a+4b-14$

$b=7$

(ii) $v_5=v_3+2v_4-7=(a+2b-7)+2(2a+5b-21)-7=5a+12b-56$

$=5a+12(7)-56$

$=5a+28$

(c) (i) Let S_n be the sum of first n terms.

$S_n = n^3 - 11n^2 + 4n$

n th term of the series

$= S_n - S_{n-1}$

$= \left(n^3 - 11n^2 + 4n\right) - \left[\left(n-1\right)^3 - 11\left(n-1\right)^2 + 4\left(n-1\right)\right]$

$= \left(n^3 - 11n^2 + 4n\right) - \left[\left(n^3 - 3n^2 + 3n - 1\right) - 11\left(n^2 - 2n + 1\right) + 4\left(n-1\right)\right]$

$= \left(n^3 - 11n^2 + 4n\right) - \left(n^3 - 14n^2 + 29n - 16\right)$

$= 3n^2 - 25n + 16$

(ii) $S_m = m^3 - 11m^2 + 4m$

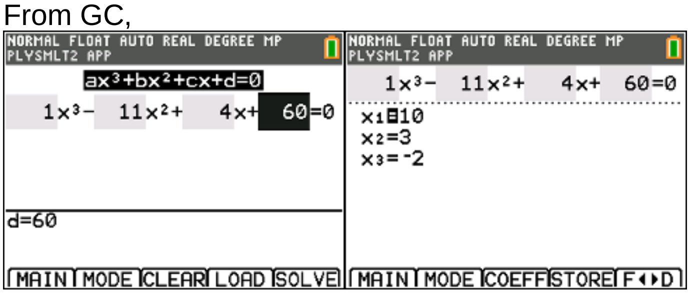
$S_3 = \left(3\right)^3 - 11\left(3\right)^2 + 4\left(3\right) = -60$

$S_m = S_3$

$m^3 - 11m^2 + 4m = -60$

$m^3 - 11m^2 + 4m + 60 = 0$

From GC,



$m = -2$ (NA) or $m = 3$ (NA) or $m = 10$

Question 3

[Ans: (i) $2x+y=39$; $a=2$, $b=1$, $c=39$ (ii) $\frac{2057}{18}$ units^2 (iii)(a) $\frac{2057}{3}$ units^2

(b) $x=\frac{1}{54}(y+3)^2+4$]

(i) $x=3t^2+2 \Rightarrow \frac{dx}{dt}=6t$

$y=6t-1 \Rightarrow \frac{dy}{dt}=6$

$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{6}{6t}=\frac{1}{t}$

At $(14,11)$,

$x=14$

$y=11 \Rightarrow 6t-1=11 \Rightarrow t=2$

$\frac{dy}{dx}=\frac{1}{2}$

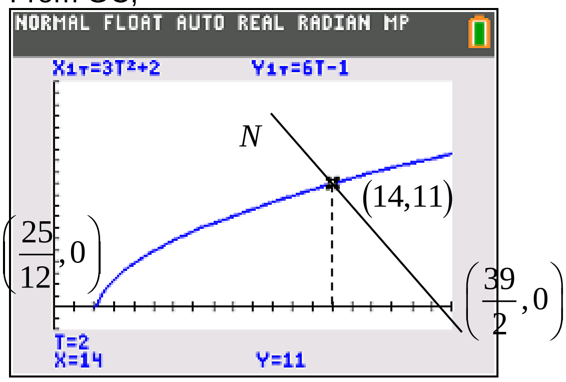
Equation of line N:

$y-11=-2(x-14)$

$2x+y=39$

By comparison, $a=2$, $b=1$, $c=39$

(ii) From GC,



Area

$$=\int_{\frac{25}{12}}^{14} y\,dx+\frac{1}{2}\left(\frac{39}{2}-14\right)(11)$$

$$=\int_{\frac{1}{6}}^{2}(6t-1)(6t)\,dt+\frac{121}{4}$$

$$=\frac{3025}{36}+\frac{121}{4}=\frac{2057}{18}$$

(iii) (a) Curve D:

$$\frac{1}{2}x = 3t^2 + 2 \Rightarrow \frac{dx}{dt} = 2(6t)$$

$$\frac{1}{3}y = 6t - 1 \Rightarrow y = 3(6t - 1)$$

Area

$$= \int_{\frac{25}{12}\times 2}^{14\times 2} y\,dx + \frac{1}{2}\left(\frac{39}{2}\times 2 - 14\times 2\right)(11\times 3)$$

$$= \int_{\frac{1}{6}}^{2} 3(6t-1)\left[2(6t)\right]dt + \frac{1}{2}\left(\frac{39}{2}\times 2 - 14\times 2\right)(11\times 3)$$

$$= \left[\int_{\frac{1}{6}}^{2}(6t-1)(6t)\,dt + \frac{1}{2}\left(\frac{39}{2} - 14\right)(11)\right](2)(3)$$

$$= \left(\frac{2057}{18}\right)(2)(3) = \frac{2057}{3}$$

(b) Curve D:

$$\frac{1}{2}x = 3t^2 + 2$$

$$\frac{1}{3}y = 6t - 1 \Rightarrow t = \frac{1}{18}y + \frac{1}{6}$$

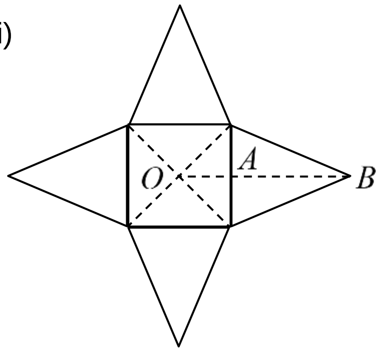
$$\frac{1}{2}x = 3\left(\frac{1}{18}y + \frac{1}{6}\right)^2 + 2$$

$$x = \frac{6}{18^2}(y+3)^2 + 4 \Rightarrow x = \frac{1}{54}(y+3)^2 + 4$$

Question 4

[Ans: (i) show (ii) $144\sqrt{5}$ units^3 (iii)(a) $a=15$ (b) square with sides $15\sqrt{2}$ cm]

(i)



$$OA=\frac{1}{2}a$$

$$AB=h=\frac{30-a}{2}=15-\frac{1}{2}a$$

$$H^2=AB^2-OA^2$$

$$=\left(15-\frac{1}{2}a\right)^2-\left(\frac{1}{2}a\right)^2$$

$$=225-2(15)\left(\frac{1}{2}a\right)+\left(\frac{1}{2}a\right)^2-\left(\frac{1}{2}a\right)^2$$

$=225-15a$ (shown)

(ii) Let V be the volume of the pyramid.

$$V=\frac{1}{3}a^2H$$

$$V^2=\frac{1}{9}a^4H^2=\frac{1}{9}a^4\left(225-15a\right)$$

$$V^2=\frac{15}{9}\left(15a^4-a^5\right)$$

$$2V\frac{dV}{da}=\frac{15}{9}\left(60a^3-5a^4\right)=\frac{75}{9}a^3\left(12-a\right)$$

Let $\frac{dV}{da}=0$,

$$\frac{75}{9}a^3\left(12-a\right)=0$$

$a=0$ (NA) or $a=12$

When $a=12$,

$$V_{\max}=\sqrt{\frac{15}{9}\left[15(12)^4-(12)^5\right]}=\sqrt{103680}=144\sqrt{5}$$

(iii) (a) Let S be the total surface area of the four triangular faces of the pyramid.

$$S=4\left(\frac{1}{2}ah\right)=2a\left(15-\frac{1}{2}a\right)=30a-a^2$$

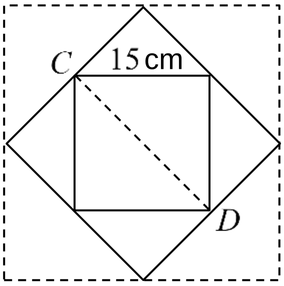
$$\frac{dS}{da}=30-2a$$

For S to be maximum,

$$\frac{dS}{da}=0$$

$$30-2a=0\Rightarrow a=15$$

(b)



$CD = \sqrt{15^2 + 15^2} = 15\sqrt{2}$

$\therefore$ The net will become a square with sides $15\sqrt{2}$ cm.

Section B

Question 5

[Ans: (i) 0, 4, 10 and 25 (ii) show; $Var(X)=\dfrac{360r^2+300r-340}{(3r+1)^2}$ (iii) $r=3$]

(i) Listing out all possible outcomes,

	0 G	2 B	5 R
0 G	-	0	0
2 B	0	4	10
5 R	0	10	25

Tina's possible scores are 0, 4, 10 and 25

(ii) Let X be Tina's score.

x	0	4	10	25
$P(X=x)$	$P(X=1)$	$\left(\frac{2r}{3r+1}\right)\left(\frac{2r-1}{3r}\right)$ $=\frac{2(2r-1)}{3(3r+1)}$	$2\left(\frac{2r}{3r+1}\right)\left(\frac{r}{3r}\right)$ $=\frac{4r}{3(3r+1)}$	$\left(\frac{r}{3r+1}\right)\left(\frac{r-1}{3r}\right)$ $=\frac{r-1}{3(3r+1)}$

$E(X)$

$$=\sum xP(X=x)=(0)P(X=1)+4\left[\frac{2(2r-1)}{3(3r+1)}\right]+10\left[\frac{4r}{3(3r+1)}\right]+25\left[\frac{r-1}{3(3r+1)}\right]$$

$$=\frac{16r-8+40r+25r-25}{3(3r+1)}$$

$$=\frac{81r-33}{3(3r+1)}=\frac{27r-11}{3r+1}$$ (shown)

$E(X^2)$

$$=\sum x^2P(X=x)=(0)^2P(X=1)+4^2\left[\frac{2(2r-1)}{3(3r+1)}\right]+10^2\left[\frac{4r}{3(3r+1)}\right]+25^2\left[\frac{r-1}{3(3r+1)}\right]$$

$$=\frac{64r-32+400r+625r-625}{3(3r+1)}$$

$$=\frac{1089r-657}{3(3r+1)}=\frac{363r-219}{3r+1}$$

$Var(X)$

$= E(X^2) - [E(X)]^2$

$= \dfrac{363r - 219}{3r+1} - \left(\dfrac{27r - 11}{3r+1}\right)^2$

$= \dfrac{(363r - 219)(3r+1) - (27r-11)^2}{(3r+1)^2}$

$= \dfrac{1089r^2 + 363r - 657r - 219 - (729r^2 - 594r + 121)}{(3r+1)^2}$

$= \dfrac{360r^2 + 300r - 340}{(3r+1)^2}$

(iii) $Var(X) = 38$

$\dfrac{360r^2 + 300r - 340}{(3r+1)^2} = 38$

$360r^2 + 300r - 340 = 38(9r^2 + 6r + 1)$

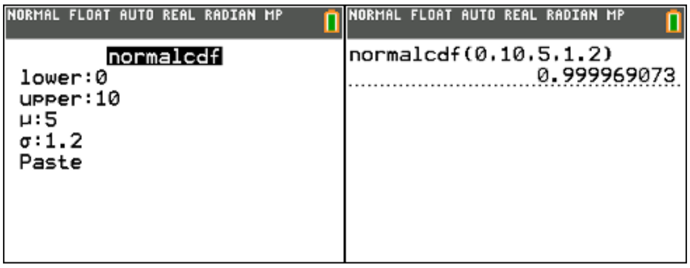
$18r^2 + 72r - 378 = 0$

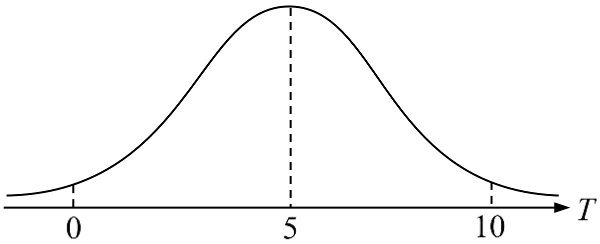
$r = -7$ (NA) or $r = 3$

Question 6

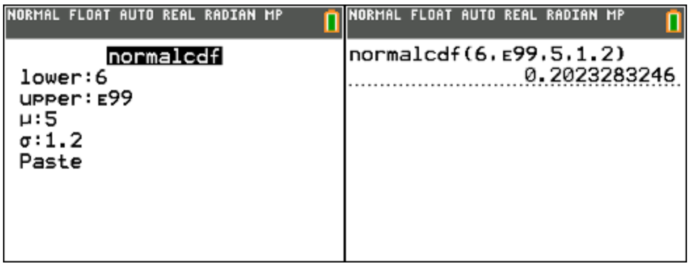
[Ans: (i) sketch (ii) 0.202 (iii) 0.108 (iv) 0.606]

(i) $P(0 < T < 10) = 0.99997$





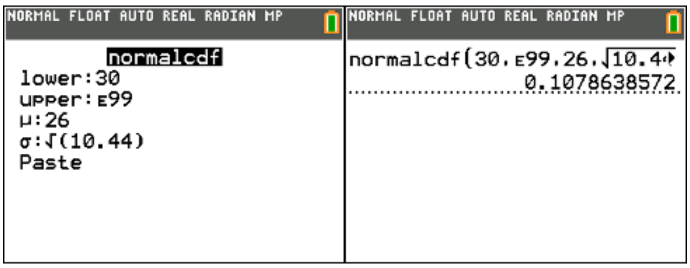
(ii) $P(T > 6) = 0.20233 \approx 0.202$



(iii) $T \sim N(5, 1.2^2)$, $W \sim N(21, 3^2)$

$$E(T+W) = E(T) + E(W) = 5 + 21 = 26$$
$$Var(T+W) = Var(T) + Var(W) = 1.2^2 + 3^2 = 10.44$$
$$T + W \sim N(26, 10.44)$$

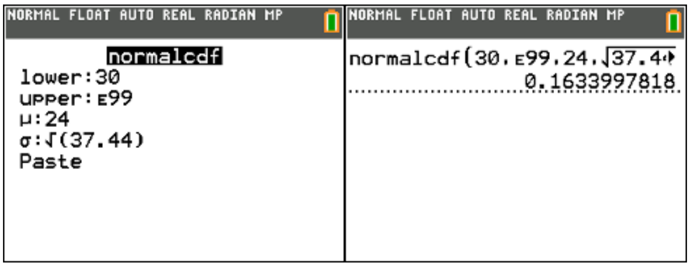
Probability that James is late for work
$= P(T + W > 30) = 0.10786 \approx 0.108$



(iv) $T \sim N\left(5, 1.2^2\right)$, $D \sim N\left(19, 6^2\right)$

$$E(T+D) = E(T) + E(D) = 5 + 19 = 24$$
$$Var(T+D) = Var(T) + Var(D) = 1.2^2 + 6^2 = 37.44$$
$$T + D \sim N(24, 37.44)$$

$$P(\text{Weather is fine} \mid \text{James is late for work})$$
$$= \frac{P(\text{Weather is fine} \cap \text{James is late for work})}{P(\text{James is late for work})}$$
$$= \frac{0.7P(T+W>30)}{0.7P(T+W>30)+0.3(T+D>30)}$$

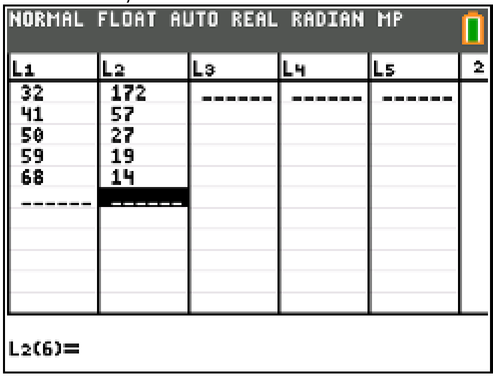


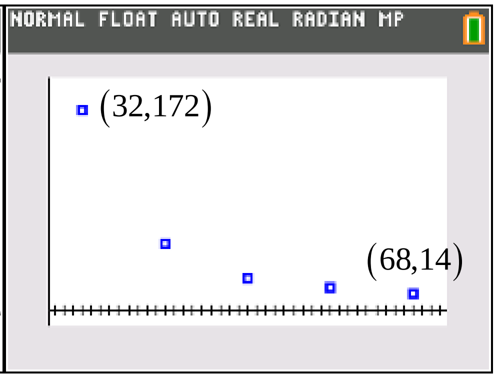
$$= \frac{(0.7)(0.10789)}{(0.7)(0.10789)+(0.3)(0.16340)} = 0.60640 \approx 0.606$$

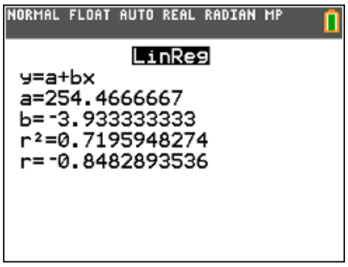
Question 7

[Ans: (i) sketch; $r=-0.848$ (ii) $a=145$, $b=9460$, $r=0.941$ (iii) does not fit (iv) $d=-145.13+\dfrac{47282}{9T+160}$]

(i) From GC,

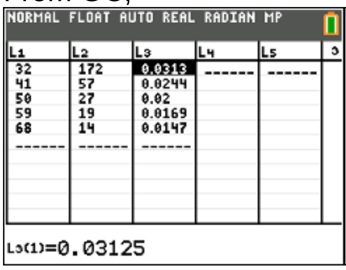


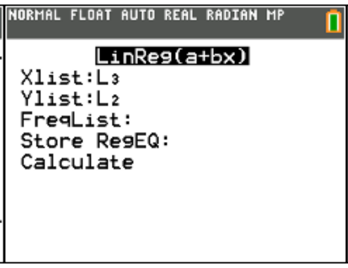


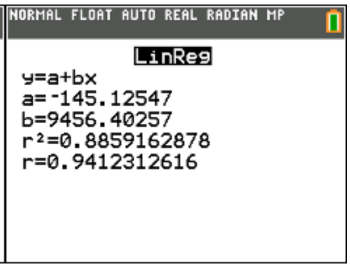


$r=-0.84829\approx-0.848$

(ii) From GC,







$a=145.13\approx145$, $b=9456.4\approx9460$

$r=0.94123\approx0.941$

(iii) $d=-145.13+\dfrac{9456.4}{t}$

When $t=86$,

$$d=-145.13+\frac{9456.4}{86}=-35.172$$

$\therefore$ Lim's model does not fit this additional data.

(iv) $T=\dfrac{5}{9}(t-32)\Rightarrow t=\dfrac{9}{5}T+32=\dfrac{9T+160}{5}$

$$d=-145.13+\frac{9456.4}{t}=-145.13+\frac{9456.4}{\frac{9T+160}{5}}$$

$$d=-145.13+\frac{47282}{9T+160}$$

Question 8

[Ans: (i) show (ii) $r = 6$]

(i) $P(R=1) = \dfrac{{}^{11}C_{11}\,{}^{17}C_1}{{}^{28}C_{12}} = 5.5881\times10^{-7}$

$P(R=2) = \dfrac{{}^{11}C_{10}\,{}^{17}C_2}{{}^{28}C_{12}} = 4.9175\times10^{-5}$

$\therefore P(R=1) < P(R=2)$ (shown)

(ii) $P(R=4) = \dfrac{{}^{11}C_8\,{}^{17+r}C_4}{{}^{28+r}C_{12}}$

$P(R=3) = \dfrac{{}^{11}C_9\,{}^{17+r}C_3}{{}^{28+r}C_{12}}$

$P(R=4) = 15P(R=3)$

$\dfrac{{}^{11}C_8\,{}^{17+r}C_4}{{}^{28+r}C_{12}} = 15\left(\dfrac{{}^{11}C_9\,{}^{17+r}C_3}{{}^{28+r}C_{12}}\right)$

$165\,{}^{17+r}C_4 = 825\,{}^{17+r}C_3$

$\dfrac{(17+r)!}{4!(17+r-4)!} = 5\left[\dfrac{(17+r)!}{3!(17+r-3)!}\right]$

$\dfrac{1}{4(13+r)!} = \dfrac{5}{(14+r)!}$

$\dfrac{1}{4(13+r)!} = \dfrac{5}{(14+r)(13+r)!}$

$\dfrac{1}{4} = \dfrac{5}{14+r}$

$14+r = 20 \Rightarrow r = 6$

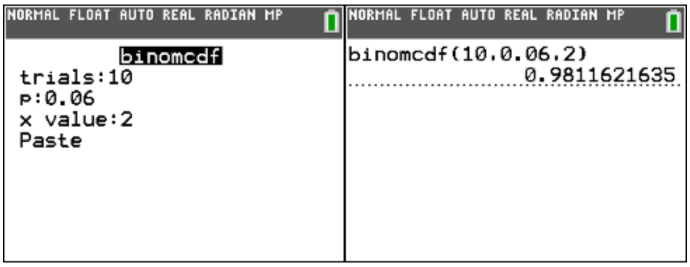
Question 9

[Ans: (i) explain (ii) 0.981 (iii) 0.0535 (iv) 0.983 (v) explain]

(i) It means that there is an equal chance for any pens in the box to be chosen.

(ii) Let X be the number of faulty pens chosen out of a random sample of 10 pens.
$X \sim B(10, 0.06)$

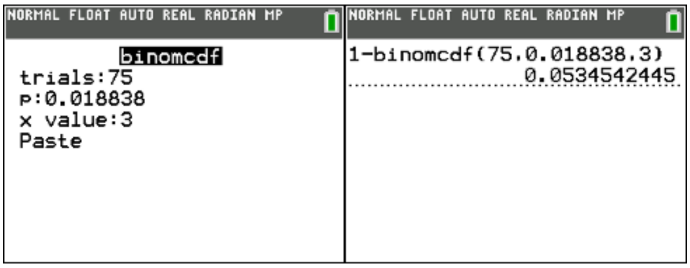
Required probability
$= P(X \leq 2) = 0.98116 \approx 0.981$



(iii) Probability of a box being rejected $= 1 - 0.981162 = 0.018838$

Let Y be the number of rejected boxes out of the 75 boxes.
$Y \sim B(75, 0.018838)$

Required probability
$= P\left(Y > \frac{5}{100}(75)\right) = P(Y > 3.75) = P(Y \geq 4)$
$= 1 - P(Y \leq 3) 0.053454 \approx 0.0535$



(iv) Let W be the number of faulty pens chosen out of a random sample of 5 pens.
$W \sim B(5, 0.06)$

Required probability
$= P(W = 0) + P(W = 1) P(W \leq 1) + P(W = 2) P(W = 0)$
$= 0.73390 + (0.23422)(0.96813) + (0.029901)(0.73390)$
$= 0.98261 \approx 0.983$

(v) The alternative testing procedure might be preferred because it produce a higher chance of a box of pens being accepted.

Question 10

[Ans: (i) sample mean <1.45 and sample mean >1.55 (ii) explain (iii) 0.254 ; 1.46×10^{-4}
(iv) p-value $=0.018211$, sufficient evidence to reject H_0]

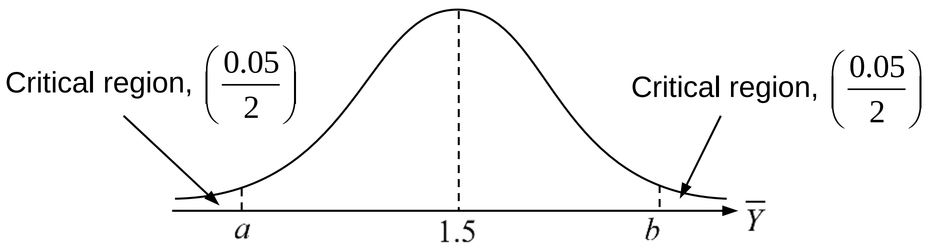
(i) Let Y be the amount of carbon in a randomly chosen round bar and μ be its mean.

$H_0:\mu=1.5$
$H_1:\mu\neq1.5$

$n=15$, $\sigma^2=0.09^2$

Test Statistics:

$\bar{Y}\sim N\left(1.5,\dfrac{0.09^2}{15}\right)$



$P\left(\bar{Y}<a\right)=\dfrac{0.05}{2}\Rightarrow a=1.4545$

$P\left(\bar{Y}>b\right)=\dfrac{0.05}{2}\Rightarrow b=1.5455$

$\therefore$ critical region: $\bar{Y}<1.45$ and $\bar{Y}>1.55$

(ii) As these mild steel bars are from a new line, there is likely not enough data to ascertain its distribution and standard deviation (*unlike in the carbon steel case where the percentage of carbon in the bar is known to be distributed normally and with standard deviation known*). Therefore by taking a large sample, $n=40$, the production manager will be able to approximate the sample mean for the amount of carbon to a normal distribution because of the Central Limit Theorem.

(iii) $\mu\approx\bar{x}=\dfrac{\sum x}{n}=\dfrac{10.16}{40}=0.254$

$$\sigma^2\approx s^2=\frac{1}{n-1}\left[\sum x^2-\frac{\left(\sum x\right)^2}{n}\right]=\frac{1}{40-1}\left(2.586342-\frac{10.16^2}{40}\right)$$

$$=1.4621\times10^{-4}\approx1.46\times10^{-4}$$

(iv) Let X be the amount of carbon in a randomly chosen flat bar, μ be its mean and σ be its standard deviation.

$H_0 : \mu = 0.25$

$H_1 : \mu > 0.25$

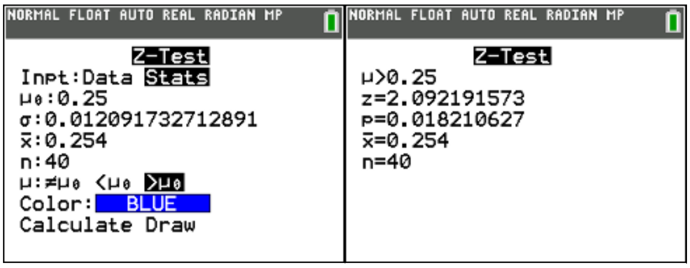
$n = 40$, $\bar{x} = 0.254$, $\sigma^2 \approx s^2 = 1.4621 \times 10^{-4}$

Test Statistics:

$\bar{X} \sim N\left(0.25, \dfrac{1.4621 \times 10^{-4}}{40}\right)$ by Central Limit Theorem, since $n = 40$ is large

$$Z = \frac{\bar{X} - 0.25}{\sqrt{\dfrac{1.4621 \times 10^{-4}}{40}}} \sim N(0,1)$$

From GC, p-value $= 0.018211$



Since p-value < 0.025, there is sufficient evidence that the mean carbon in the flat bars is more than 0.25% at $2\frac{1}{2}\%$ level of significance.