A-LEVEL H2 MATH 2020 – PAPER 1

Question 1

[Ans: (i) $\begin{pmatrix}1\\-1\\3\end{pmatrix}$ (ii) $49.2°$]

(i) $\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}1\\-5\\-2\end{pmatrix}=\begin{pmatrix}-2\\2\\-6\end{pmatrix}=-2\begin{pmatrix}1\\-1\\3\end{pmatrix}$

A vector normal to $\pi_1=\begin{pmatrix}1\\-1\\3\end{pmatrix}$

(ii) $\pi_2:\underset{\sim}{r}\cdot\begin{pmatrix}4\\5\\-6\end{pmatrix}=0$

Let θ be the acute angle between π_1 and π_2.

$$\left|\begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}4\\5\\-6\end{pmatrix}\right|=\left|\begin{pmatrix}1\\-1\\3\end{pmatrix}\right|\left|\begin{pmatrix}4\\5\\-6\end{pmatrix}\right|\cos\theta$$

$$|4-5-18|=\sqrt{1^2+1^2+3^2}\sqrt{4^2+5^2+6^2}\cos\theta$$

$$\cos\theta=\frac{|-19|}{\sqrt{11}\sqrt{77}}$$

$$\theta=\cos^{-1}\left(\frac{19}{\sqrt{11}\sqrt{77}}\right)=49.2°$$

Question 2

[Ans: $5x+9y=14$]

$$\frac{d}{dx}\left(\frac{x^2}{1+x^2}+\frac{y^2}{1+y^2}\right)=\frac{d}{dx}\left(x^3y^5\right)$$

$$\frac{\left(1+x^2\right)(2x)-x^2(2x)}{\left(1+x^2\right)^2}+\left[\frac{\left(1+y^2\right)(2y)-y^2(2y)}{\left(1+y^2\right)^2}\right]\frac{dy}{dx}=x^3\left(5y^4\frac{dy}{dx}\right)+3x^2y^5$$

$$\frac{\left(1+x^2-x^2\right)(2x)}{\left(1+x^2\right)^2}+\left[\frac{\left(1+y^2-y^2\right)(2y)}{\left(1+y^2\right)^2}\right]\frac{dy}{dx}=5x^3y^4\frac{dy}{dx}+3x^2y^5$$

$$\frac{2x}{\left(1+x^2\right)^2}+\frac{2y}{\left(1+y^2\right)^2}\frac{dy}{dx}=5x^3y^4\frac{dy}{dx}+3x^2y^5$$

At $(1,1)$,

$$\frac{2}{(1+1)^2}+\frac{2}{(1+1)^2}\frac{dy}{dx}=5(1)(1)\frac{dy}{dx}+3(1)(1)$$

$$\frac{1}{2}+\frac{1}{2}\frac{dy}{dx}=5\frac{dy}{dx}+3$$

$$\frac{9}{2}\frac{dy}{dx}=-\frac{5}{2}\Rightarrow\frac{dy}{dx}=-\frac{5}{9}$$

Equation of tangent:

$$y-1=-\frac{5}{9}(x-1)$$

$$-9y+9=5x-5\Rightarrow 5x+9y=14$$

Question 3

[Ans: (i) show; $k=-9$ (ii) $f(x)=3x-\frac{9}{2}x^2+\frac{9}{2}x^3+\ldots$]

(i) $f(x)=\ln(1+\sin 3x)$

$$f'(x)=\frac{1}{1+\sin 3x}(3\cos 3x)=\frac{3\cos 3x}{1+\sin 3x}$$

$$f''(x)=\frac{(1+\sin 3x)[3(-3\sin 3x)]-3\cos 3x(3\cos 3x)}{(1+\sin 3x)^2}$$

$$=\frac{-9\sin 3x-9\sin^2 3x-9\cos^2 3x}{(1+\sin 3x)^2}$$

$$=\frac{-9-9\sin 3x}{(1+\sin 3x)^2}$$

$$=\frac{-9(1+\sin 3x)}{(1+\sin 3x)^2}=-\frac{9}{1+\sin 3x},\ k=-9$$

(ii) $f(0)=\ln(1+0)=0$

$$f'(0)=\frac{3(1)}{1+0}=3$$

$$f''(0)=-\frac{9}{1+0}=-9$$

$$f'''(x)=-9(-1)(1+\sin 3x)^{-2}(3\cos 3x)=\frac{27\cos 3x}{(1+\sin 3x)^2}$$

$$f'''(0)=\frac{27(1)}{(1+0)^2}=27$$

$$f(x)=0+(3)x+\frac{(-9)}{2!}x^2+\frac{27}{3!}x^3+\ldots$$

$$=3x-\frac{9}{2}x^2+\frac{9}{2}x^3+\ldots$$

Question 4

[Ans: (i) $\frac{1}{\sqrt{2}}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$ (ii) $z_4=\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$ or $z_4=\sqrt{2}\left[\cos\left(-\frac{11\pi}{12}\right)+i\sin\left(-\frac{11\pi}{12}\right)\right]$]

(i) $|z_1|=\sqrt{1^2+\left(\sqrt{3}\right)^2}=2$; $\arg(z_1)=\tan^{-1}\frac{\sqrt{3}}{1}=\frac{\pi}{3}$

$\therefore z_1=2e^{i\frac{\pi}{3}}$

$|z_2|=\sqrt{1^2+1^2}=\sqrt{2}$; $\arg(z_2)=-\tan^{-1}\frac{1}{1}=-\frac{\pi}{4}$

$\therefore z_2=\sqrt{2}e^{-i\frac{\pi}{4}}$

$\therefore z_3=2e^{i\frac{\pi}{6}}$

$$\frac{z_1}{z_2z_3}=\frac{2e^{i\frac{\pi}{3}}}{\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)\left(2e^{i\frac{\pi}{6}}\right)}=\frac{1}{\sqrt{2}}e^{i\left(\frac{\pi}{3}+\frac{\pi}{4}-\frac{\pi}{6}\right)}=\frac{1}{\sqrt{2}}e^{i\frac{5\pi}{12}}$$

$$=\frac{1}{\sqrt{2}}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$$

(ii) Let $z_4=re^{i\theta}$

$$\frac{z_1z_4}{z_2z_3}=\left(\frac{z_1}{z_2z_3}\right)z_4=\left(\frac{1}{\sqrt{2}}e^{i\frac{5\pi}{12}}\right)\left(re^{i\theta}\right)$$

$$=\frac{r}{\sqrt{2}}e^{i\left(\frac{5\pi}{12}+\theta\right)}=\frac{r}{\sqrt{2}}\left[\cos\left(\frac{5\pi}{12}+\theta\right)+i\sin\left(\frac{5\pi}{12}+\theta\right)\right]$$

$$\left|\frac{z_1z_4}{z_2z_3}\right|=1\Rightarrow\frac{r}{\sqrt{2}}=1\Rightarrow r=\sqrt{2}$$

$$\mathrm{Re}\left(\frac{z_1z_4}{z_2z_3}\right)=0$$

$$\cos\left(\frac{5\pi}{12}+\theta\right)=0$$

$$\theta=\frac{\pi}{2}-\frac{5\pi}{12},-\frac{\pi}{2}-\frac{5\pi}{12}=\frac{\pi}{12},-\frac{11\pi}{12}$$

$\therefore z_4=\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$ or $z_4=\sqrt{2}\left[\cos\left(-\frac{11\pi}{12}\right)+i\sin\left(-\frac{11\pi}{12}\right)\right]$

Queston 5

[Ans: (a) $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel to each other

(b)(i) The set of all possible positions of the point R consists of all the points lying on a line that passes through the point P and parallel to vector $\underset{\sim}{q}$.

(b)(ii) $3x-5y+2z=-5$; The set of all possible positions of the point R consists of all the points lying on a plane that passes through the point P and normal to vector $\begin{pmatrix}3\\-5\\2\end{pmatrix}$.]

(a) $\underset{\sim}{a}\times\underset{\sim}{b}=\underset{\sim}{b}\times\underset{\sim}{a}$

$\underset{\sim}{a}\times\underset{\sim}{b}-\underset{\sim}{b}\times\underset{\sim}{a}=\underset{\sim}{0}$

$\underset{\sim}{a}\times\underset{\sim}{b}+\underset{\sim}{a}\times\underset{\sim}{b}=\underset{\sim}{0}$

$2\underset{\sim}{a}\times\underset{\sim}{b}=\underset{\sim}{0}\Rightarrow\underset{\sim}{a}\times\underset{\sim}{b}=\underset{\sim}{0}$

$\therefore\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel to each other.

(b) (i) $\left(\underset{\sim}{r}-\underset{\sim}{p}\right)\times\underset{\sim}{q}=\underset{\sim}{0}\Rightarrow\underset{\sim}{r}-\underset{\sim}{p}=\lambda\underset{\sim}{q},\lambda\in\mathbb{R}$

$\therefore\underset{\sim}{r}=\underset{\sim}{p}+\lambda\underset{\sim}{q},\lambda\in\mathbb{R}$

$\therefore$ The set of all possible positions of the point R consists of all the points lying on a line that passes through the point P and parallel to vector $\underset{\sim}{q}$.

(ii) $\left(\underset{\sim}{r}-\underset{\sim}{p}\right)\cdot\underset{\sim}{q}=\underset{\sim}{0}$

$$\left(\begin{pmatrix}x\\y\\z\end{pmatrix}-\begin{pmatrix}-1\\2\\4\end{pmatrix}\right)\cdot\begin{pmatrix}3\\-5\\2\end{pmatrix}=0$$

$$\begin{pmatrix}x+1\\y-2\\z-4\end{pmatrix}\cdot\begin{pmatrix}3\\-5\\2\end{pmatrix}=0$$

$3x+3-5y+10+2z-8=0$

$3x-5y+2z=-5$

The set of all possible positions of the point R consists of all the points lying on a plane that passes through the point P and normal to vector $\begin{pmatrix}3\\-5\\2\end{pmatrix}$.

Question 6

[Ans: $t=-8$, $k=2$; $-\frac{2}{5}+\frac{6}{5}i$]

Let $f(z)=z^2(2+i)-8iz+t$.

$f(k+ki)=0$

$(k+ki)^2(2+i)-8i(k+ki)+t=0$

$k^2(1+i)^2(2+i)-8ki+8k+t=0$

$k^2(-2+4i)-8ki+8k+t=0$

$(-2k^2+8k+t)+(4k^2-8k)i=0$

$\therefore 4k^2-8k=0$

$4k(k-2)=0$

$k=0$ (NA) or $k=2$

$-2k^2+8k+t=0$

$t=2k^2-8k=2(2)^2-8(2)$

$=-8$

$z^2(2+i)-8iz-8=0$

$$z=\frac{-(-8i)\pm\sqrt{(-8i)^2-4(2+i)(-8)}}{2(2+i)}$$

$$=\frac{8i\pm(4+4i)}{2(2+i)}$$

$z=\frac{8i-(4+4i)}{2(2+i)}$ or $z=\frac{8i+(4+4i)}{2(2+i)}$

$=-\frac{2}{5}+\frac{6}{5}i$ $\quad$ $=2+2i$

$\therefore$ the other root of the equation is $-\frac{2}{5}+\frac{6}{5}i$.

Question 7

[Ans: (i) $2x+\frac{1}{4}\cos 4x+c$ (ii) $\frac{1}{8}\pi(2\pi+1)$ (iii) $\frac{9}{4}\pi$]

(i) $\int f(x)dx=\int 2-\sin 4x dx=2x+\frac{1}{4}\cos 4x+c$

(ii) $\int_0^{\frac{1}{2}\pi} xf(x)dx$

$$u=x,\ \frac{dv}{dx}=f(x)$$

$$\frac{du}{dx}=1,\ v=2x+\frac{1}{4}\cos 4x$$

$$=\left[(x)\left(2x+\frac{1}{4}\cos 4x\right)\right]_0^{\frac{1}{2}\pi}-\int_0^{\frac{1}{2}\pi} 2x+\frac{1}{4}\cos 4x dx$$

$$=\left[\left(\frac{1}{2}\pi\right)\left(\pi+\frac{1}{4}\cos 2\pi\right)-(0)\right]-\left[x^2+\frac{1}{16}\sin 4x\right]_0^{\frac{1}{2}\pi}$$

$$=\frac{1}{2}\pi^2+\frac{1}{8}\pi-\left[\left(\frac{1}{4}\pi^2+0\right)-(0)\right]$$

$$=\frac{1}{4}\pi^2+\frac{1}{8}\pi=\frac{1}{8}\pi(2\pi+1)$$

(iii) $\int_0^{\frac{1}{2}\pi}(f(x))^2 dx$

$$=\int_0^{\frac{1}{2}\pi}(2-\sin 4x)^2 dx$$

$$=\int_0^{\frac{1}{2}\pi}4-4\sin 4x+\sin^2 4x dx$$

$$=\int_0^{\frac{1}{2}\pi}4-4\sin 4x+\frac{1-\cos 8x}{2}dx=\int_0^{\frac{1}{2}\pi}\frac{9}{2}-4\sin 4x-\frac{1}{2}\cos 8x dx$$

$$=\left[\frac{9}{2}x+\cos 4x-\frac{1}{16}\sin 8x\right]_0^{\frac{1}{2}\pi}$$

$$=\left[\frac{9}{4}\pi+\cos 2\pi-\frac{1}{16}\sin 4\pi\right]-\left[0+\cos 0-\frac{1}{16}\sin 0\right]$$

$$=\left(\frac{9}{4}\pi+1\right)-(1)=\frac{9}{4}\pi$$

Question 8

[Ans: (a)(i) $\frac{95}{2}$ (ii) $\frac{3345}{2}$ (i) 20 (ii) 18]

(a) (i) Let d be the common difference of the arithmetic series.

5th term $=10$

$4+(5-1)d=10 \Rightarrow d=\frac{3}{2}$

$\therefore 30$th term of this series $=4+(30-1)\left(\frac{3}{2}\right)=\frac{95}{2}$

(ii) Sum of first 20 terms $=\frac{20}{2}\left[2(4)+(20-1)\left(\frac{3}{2}\right)\right]=365$

Sum of first 50 terms $=\frac{50}{2}\left[2(4)+(50-1)\left(\frac{3}{2}\right)\right]=\frac{4075}{2}$

Required sum $=\frac{4075}{2}-365=\frac{3345}{2}$

(b) (i) Let r be the common ratio of the geometric series.

5th term $=1.6384$

$4r^{5-1}=1.6384 \Rightarrow r^4=0.4096 \Rightarrow r=0.8 (\because r>0)$

$\therefore$ Sum to infinity $=\frac{4}{1-0.8}=20$

(ii) Sum of first n terms >19.6

$\frac{4(1-0.8^n)}{1-0.8}>19.6$

$1-0.8^n>19.6\left(\frac{0.2}{4}\right)$

$0.8^n<1-19.6\left(\frac{0.2}{4}\right) \Rightarrow 0.8^n<0.02$ (shown)

$n\ln 0.8<\ln 0.02$

$n>\frac{\ln 0.02}{\ln 0.8}$

$n>17.531$

$\therefore$ smallest value of $n=18$

Question 9

[Ans: (i) explain (ii) $\left\{k\in\mathbb{R}:0<k\le\frac{9}{2}\right\}$ (iii) sketch (iv) $\frac{1}{2}\pi-\frac{4}{3}\ln 2$]

(i) Let the gradient of a line be m_1 and the angle it makes with the positive x-axis be θ_1.
Let the gradient of another line be m_2 and the angle it makes with the positive x-axis be θ_2.

Let $m_1=2\Rightarrow\tan\theta_1=2\Rightarrow\theta_1=\tan^{-1}(2)$

Let $m_2=-\frac{1}{2}\Rightarrow\tan\theta_2=-\frac{1}{2}\Rightarrow\theta_2=\pi-\tan^{-1}\left(\frac{1}{2}\right)=\pi+\tan^{-1}\left(-\frac{1}{2}\right)$

Since $m_1m_2=(2)\left(-\frac{1}{2}\right)=-1$, the two lines are perpendicular to each other.

$\therefore\theta_2-\theta_1=\frac{1}{2}\pi$

$\Rightarrow\left[\pi+\tan^{-1}\left(-\frac{1}{2}\right)\right]-\tan^{-1}(2)=\frac{1}{2}\pi$

$\therefore\tan^{-1}(2)-\tan^{-1}\left(-\frac{1}{2}\right)=\frac{1}{2}\pi$

(ii) $y=\frac{1}{x^2+1}$ (1) $\qquad$ $y=\frac{k}{3x+4}$ (2)

(1) = (2)

$\frac{1}{x^2+1}=\frac{k}{3x+4}$

$3x+4=kx^2+k$

$kx^2-3x+(k-4)=0$

Let Discriminant ≥ 0

$(-3)^2-4k(k-4)\ge 0$

$4k^2-16k-9\le 0$

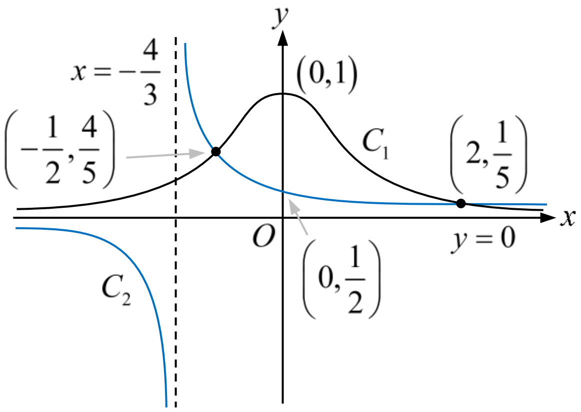
$(2k+1)(2k-9)\le 0$

$-\frac{1}{2}\le k\le\frac{9}{2}$

Since $k>0$,

$\therefore\left\{k\in\mathbb{R}:0<k\le\frac{9}{2}\right\}$

(iii) $C_1 : y = \dfrac{1}{x^2+1}$, $C_2 : y = \dfrac{2}{3x+4}$



(iv) Area

$$= \int_{-\frac{1}{2}}^{2} \frac{1}{x^2+1} - \frac{2}{3x+4}\,dx$$

$$= \left[\tan^{-1} x - \frac{2}{3}\ln|3x+4| \right]_{-\frac{1}{2}}^{2}$$

$$= \left[\tan^{-1}(2) - \frac{2}{3}\ln 10 \right] - \left[\tan^{-1}\left(-\frac{1}{2}\right) - \frac{2}{3}\ln\frac{5}{2} \right]$$

$$= \left[\tan^{-1}(2) - \tan^{-1}\left(-\frac{1}{2}\right) \right] + \left(\frac{2}{3}\ln\frac{5}{2} - \frac{2}{3}\ln 10 \right)$$

$$= \frac{1}{2}\pi + \frac{2}{3}\ln\left(\frac{5/2}{10} \right) = \frac{1}{2}\pi + \frac{2}{3}\ln\left(\frac{1}{4}\right) = \frac{1}{2}\pi - \frac{4}{3}\ln 2$$

Question 10

[Ans: (i) $\frac{dP}{dt} = -0.03P$ (ii) $P = Ce^{-0.03t}$; the number of sheep will go to zero

(iii) $\frac{dP}{dt} = n - 0.03P$ (iv) $P = \frac{100}{3}n + De^{-0.03t}$ (v) 15]

(i) $\frac{dP}{dt} = -0.03P$

(ii) $\frac{1}{P}\frac{dP}{dt} = -0.03$

$\int \frac{1}{P}\, dP = -0.03\int dt$

$\ln|P| = -0.03t + A$

$|P| = e^{-0.03t+A} = e^{A}e^{-0.03t}$

$P = e^{A}e^{-0.03t}$

$\quad = Be^{-0.03t}, \ B = e^{A}$

When $t \to \infty$, $P \to 0$

$\therefore$ if this situation continues over many years, the number of sheep will go to zero.

(iii) $\frac{dP}{dt} = n - 0.03P$

(iv) $\frac{1}{n-0.03P}\frac{dP}{dt} = 1$

$\int \frac{1}{n-0.03P}\, dP = \int dt$

$\frac{1}{-0.03}\ln|n-0.03P| = t + C$

$\ln|n-0.03P| = -0.03t - 0.03C$

$|n-0.03P| = e^{-0.03t-0.03C} = e^{-0.03C}e^{-0.03t}$

$n - 0.03P = \pm e^{-0.03C}e^{-0.03t}$

$n - 0.03P = \pm e^{-0.03C}e^{-0.03t}$

$P = \frac{100}{3}n \pm \frac{100}{3}e^{-0.03C}e^{-0.03t}$

$\quad = \frac{100}{3}n + De^{-0.03t}, \ D = \pm\frac{100}{3}e^{-0.03C}$

(v) When $t \to \infty$, $P \to 500$

$\therefore \frac{100}{3}n = 500 \Rightarrow n = 15$

Question 11

[Ans: (i) show (ii) $x=\sqrt{4a+a^2}$; $\tan\theta=\dfrac{2}{\sqrt{4a+a^2}}$ (iii) explain (iv) show; $\dfrac{\pi}{4}$

(v) $0.0801\le\theta<\dfrac{\pi}{2}$]

(i) $\tan\theta$

$$=\tan\left(AKD-AKC\right)$$

$$=\frac{\tan AKD-\tan AKC}{1+\left(\tan AKD\right)\left(\tan AKC\right)}$$

$$=\frac{\dfrac{4+a}{x}-\dfrac{a}{x}}{1+\left(\dfrac{4+a}{x}\right)\left(\dfrac{a}{x}\right)}$$

$$=\frac{\dfrac{4+a-a}{x}}{\dfrac{x^2+4a+a^2}{x^2}}=\frac{4x}{x^2+4a+a^2}$$ (shown)

(ii) Let $y=\tan\theta=\dfrac{4x}{x^2+4a+a^2}$

$$\frac{dy}{dx}=\frac{\left(x^2+4a+a^2\right)(4)-(4x)(2x)}{\left(x^2+4a+a^2\right)^2}=\frac{-4x^2+16a+4a^2}{\left(x^2+4a+a^2\right)^2}$$

Let $\dfrac{dy}{dx}=0$

$$\frac{-4x^2+16a+4a^2}{\left(x^2+4a+a^2\right)^2}=0$$

$$-4x^2+16a+4a^2=0$$

$$x^2=4a+a^2\Rightarrow x=\sqrt{4a+a^2}$$

When $x=\sqrt{4a+a^2}$,

$$\tan\theta=\frac{4\sqrt{4a+a^2}}{\left(4a+a^2\right)+4a+a^2}=\frac{4\sqrt{4a+a^2}}{2\left(4a+a^2\right)}$$

$$=\frac{2}{\sqrt{4a+a^2}}$$

(iii) At the optimal point, although $\tan\theta$ is maximized, a player may decide not to take the kick if the physical distance to get the ball between C and D is perceived to be beyond what he/she can likely achieve.

(iv) When θ is the optimal angle, $x = \sqrt{4a + a^2}$,

$$\tan KDA = \frac{x}{4+a} = \frac{\sqrt{4a+a^2}}{4+a} = \frac{\sqrt{a}\sqrt{4+a}}{4+a}$$

$$= \frac{\sqrt{a}}{\sqrt{4+a}} = \sqrt{\frac{a}{4+a}} \text{ (shown)}$$

When a is much greater than 4,

$$\tan KDA \approx \sqrt{\frac{a}{a}} = 1$$

$$KDA \approx \frac{\pi}{4}$$

(v) At optimal angle,

$$\tan\theta = \frac{2}{\sqrt{4a+a^2}}$$

θ is the greatest when $a = 0$ (A coincides with C),

$\tan\theta$ is undefined

$$\theta \to \frac{\pi}{2}$$

θ is the least when $a = \frac{50-4}{2} = 23$ (A coincides with X),

$$\tan\theta = \frac{2}{\sqrt{4(23)+(23)^2}}$$

$$\theta = \tan^{-1}\frac{2}{\sqrt{621}} = 0.080086$$

$$\therefore 0.0801 \le \theta < \frac{\pi}{2}$$