O-LEVEL A-MATHS 2020 – PAPER 2

Question 1

[Ans: (i) 1.25 (ii) largest is 80 when $\theta = 2.09$; smallest is 15; $\theta = 0.519$]

(i) Let $7\cos\theta + 4\sin\theta = R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$R\cos\alpha = 7$ (A)

$R\sin\alpha = 4$ (B)

$(\text{A})^2 + (\text{B})^2$ $\quad R^2 = 7^2 + 4^2 \Rightarrow R = \sqrt{65}$

$\dfrac{(\text{B})}{(\text{A})}$ $\quad \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{4}{7} \Rightarrow \tan\alpha = \dfrac{4}{7} \Rightarrow \alpha = \tan^{-1}\dfrac{4}{7} = 0.51915$

$7\cos\theta + 4\sin\theta = \sqrt{65}\cos(\theta - 0.51915)$

$7\cos\theta + 4\sin\theta = 6$

$\sqrt{65}\cos(\theta - 0.51915) = 6$

$\cos(\theta - 0.51915) = \dfrac{6}{\sqrt{65}}$

Basic $\measuredangle = \cos^{-1}\dfrac{6}{\sqrt{65}}$

$\theta - 0.51915 = \cos^{-1}\dfrac{6}{\sqrt{65}}$

$\theta = \cos^{-1}\dfrac{6}{\sqrt{65}} + 0.51915 = 1.25$

(ii) $80 - (7\cos\theta + 4\sin\theta)^2$

$= 80 - \left[\sqrt{65}\cos(\theta - 0.51915)\right]^2 = 80 - 65\cos^2(\theta - 0.51915)$

When $80 - (7\cos\theta + 4\sin\theta)^2$ is the largest,

$80 - (7\cos\theta + 4\sin\theta)^2 = 80 - 65(0)^2 = 80$

$\cos(\theta - 0.51915) = 0$

$\theta - 0.51915 = \dfrac{\pi}{2} \Rightarrow \theta = 2.09$

When $80 - (7\cos\theta + 4\sin\theta)^2$ is the smallest,

$80 - (7\cos\theta + 4\sin\theta)^2 = 80 - 65(\pm 1)^2 = 15$

$\cos(\theta - 0.51915) = \pm 1$

$\theta - 0.51915 = 0, \pi$

$\theta = 0.519$ or 3.66 (NA)

Question 2

[Ans: (a) $x=-3$, $y=11$ or $x=\frac{9}{2}$, $y=\frac{67}{2}$ (b) -4 (c) $c=-1$ or $c=5$]

(a) $y=2x^2-7$ (1)
$y=3x+20$ (2)

(1) = (2)
$2x^2-7=3x+20$
$2x^2-3x-27=0$
$(x+3)(2x-9)=0$
$x=-3$ or $x=\frac{9}{2}$

When $x=-3$,
$y=3(-3)+20=11$

When $x=\frac{9}{2}$,
$y=3\left(\frac{9}{2}\right)+20=\frac{67}{2}$

(b) For ax^2+5x-2 to be negative for all x,
$a<0$ and Discriminant <0
$5^2-4a(-2)<0$
$8a<-25$
$a<-\frac{25}{8}=-3\frac{1}{8}$

$\therefore$ greatest value of $a=-4$

(c) $y=4x+c$ (1)
$y=x^2+cx+\frac{21}{4}$ (2)

(1) = (2)
$4x+c=x^2+cx+\frac{21}{4}$
$x^2+(c-4)x+\frac{21}{4}-c=0$
Discriminant $=0$
$(c-4)^2-4(1)\left(\frac{21}{4}-c\right)=0$
$c^2-8c+16-21+4c=0$
$c^2-4c-5=0$
$(c+1)(c-5)=0$
$c=-1$ or $c=5$

Question 3

[Ans: (i) explain (ii) -3024]

(i) $(r+1)$ th term of $\left(\frac{3}{x^2}+x\right)^8$

$$=\binom{8}{r}\left(\frac{3}{x^2}\right)^{8-r}\left(x^r\right)$$

$$=\binom{8}{r}\left(3^{8-r}\right)\left(x^{-16+2r}\right)\left(x^r\right)$$

$$=\binom{8}{r}\left(3^{8-r}\right)\left(x^{-16+3r}\right)$$

If $(r+1)$ th term of $\left(\frac{3}{x^2}+x\right)^8$ is independent of x,

$$-16+3r=0 \Rightarrow r=\frac{16}{3}\notin\mathbb{Z}$$

Since r has to be an integer value, $\therefore$ there will be no terms that are independent of x $\Rightarrow$ every term is dependent on x.

(ii) $\binom{8}{r}\left(3^{8-r}\right)\left(x^{-16+3r}\right)\times(-2x)=-2\binom{8}{r}\left(3^{8-r}\right)\left(x^{-15+3r}\right)$

Let $-15+3r=0 \Rightarrow r=5$

Term independent of x

$$=-2\binom{8}{5}\left(3^{8-5}\right)x^0=-3024$$

Question 4

[Ans: Prove]

$\angle BAS=\angle ACB$ (Tangent-chord th.)

$\angle BDE+\angle ACB=180°$ (Opp. $\angle$ s in a cyclic quadrilateral)
$\angle BDE+\angle BAS=180°$

$\angle BDE+\angle ADE=180°$ ($\angle$ s on a straight line)

$\therefore \angle BAS=\angle ADE$
$\Rightarrow DE // ST$ (Alt. $\angle$ s)

Queston 5

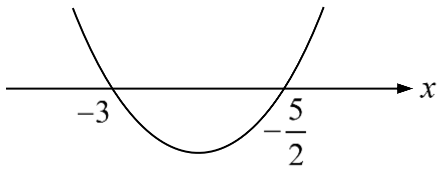
[Ans: (a) Representation of $x \le -3$ or $x \ge -\frac{5}{2}$ (b)(i) sketch (ii) show]

(a) $15(1+2x) \ge x(19-2x)$

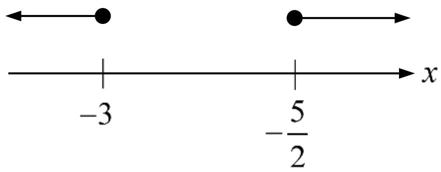
$15+30x \ge 19x-2x^2$

$2x^2+11x+15 \ge 0$

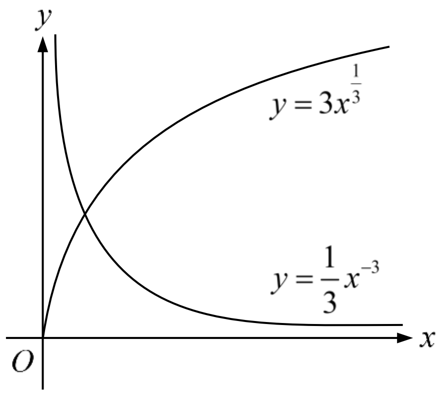
$(x+3)(2x+5) \ge 0$



$x \le -3$ or $x \ge -\frac{5}{2}$



(b) (i)



(ii) Let the point of intersection,

$3x^{\frac{1}{3}} = \frac{1}{3}x^{-3}$

$x^{\frac{1}{3}}x^3 = \frac{1}{9}$

$x^{\frac{10}{3}} = \frac{1}{9}$

$\left(x^{\frac{10}{3}}\right)^3 = \left(\frac{1}{9}\right)^3$

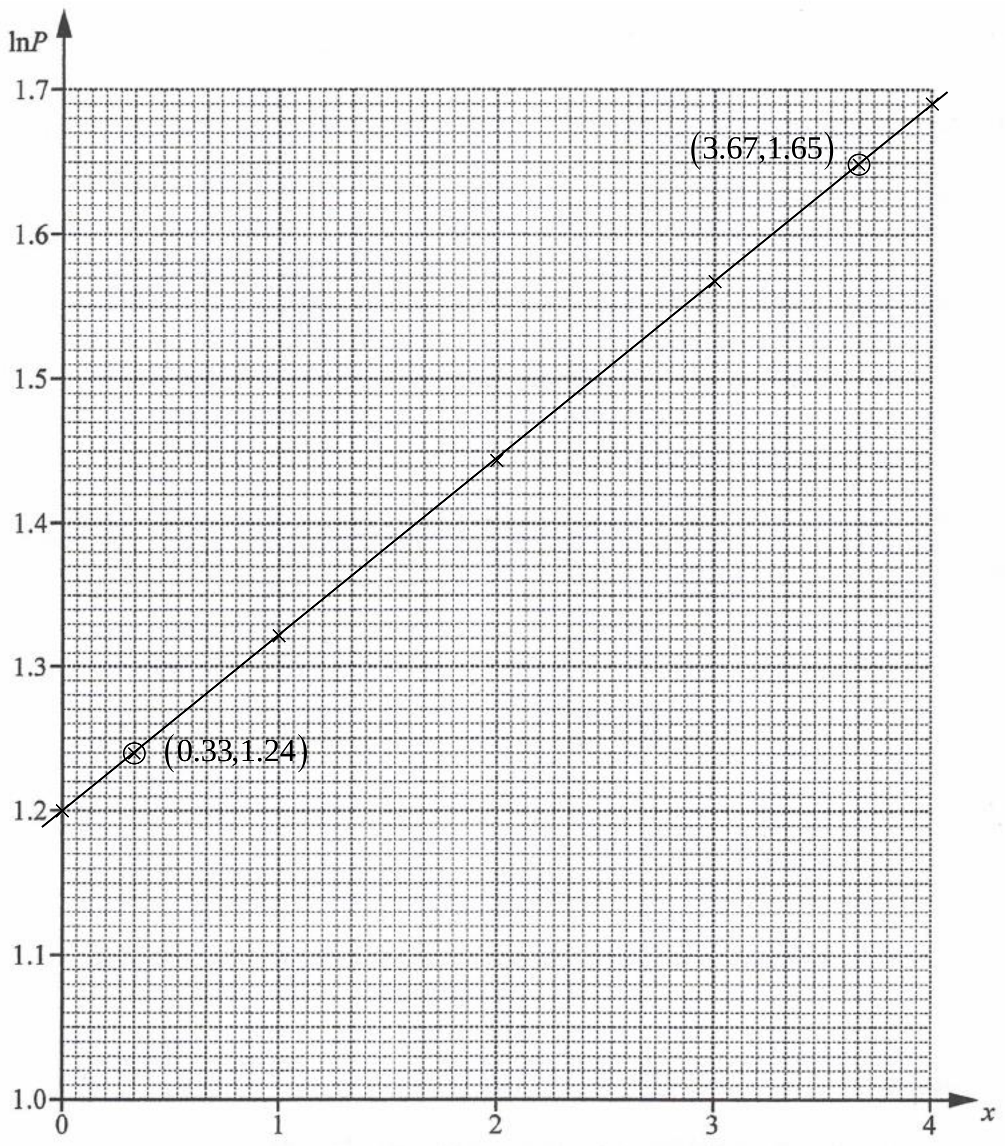
$x^{10} = \frac{1}{729}$ (shown)

Question 6

[Ans: (i) plot & draw (ii) gradient ≈ 0.123; $A = 3.32$, $k = 0.123$ (iii) 2030]

(i)

Year	1995	2000	2005	2010	2015
x	0	1	2	3	4
$\ln P$	1.200	1.322	1.445	1.569	1.692



(ii) Gradient

$$= \frac{1.65 - 1.24}{3.67 - 0.33} = 0.12275 \approx 0.123 \text{ (to 3 s.f.)}$$

$$\ln P = (0.12275)x + 1.2$$

$$P = e^{0.12275x + 1.2}$$

$$P = e^{1.2}e^{0.12275x}$$

$$P = 3.3201e^{0.12275x}$$

$$P = 3.32e^{0.123x} \text{ (to 3 s.f.)}$$

(iii) Let $P=8$

$$3.3201e^{0.12275x}=8$$

$$e^{0.12275x}=\frac{8}{3.3201}$$

$$0.12275x=\ln\left(\frac{8}{3.3201}\right)$$

$$x=\frac{1}{0.12275}\ln\left(\frac{8}{3.3201}\right)=7.1645$$

The first year which the population first exceeds 8 million

$=1995+5(7)=2030$

Question 7

[Ans: (i) show (ii) $k=\frac{1}{8}$ (iii) 0 ; explain]

(i) $\frac{d}{dx}\left\{x(3x-5)^{\frac{5}{3}}\right\}$

$=x\left[\left(\frac{5}{3}\right)(3x-5)^{\frac{2}{3}}(3)\right]+(1)(3x-5)^{\frac{5}{3}}$

$=5x(3x-5)^{\frac{2}{3}}+(3x-5)^{1+\frac{2}{3}}$

$=5x(3x-5)^{\frac{2}{3}}+(3x-5)(3x-5)^{\frac{2}{3}}$

$=(5x+3x-5)(3x-5)^{\frac{2}{3}}$

$=(8x-5)(3x-5)^{\frac{2}{3}}$ (shown)

(ii) $\frac{d}{dx}\left\{x(3x-5)^{\frac{5}{3}}\right\}=(8x-5)(3x-5)^{\frac{2}{3}}$

$x(3x-5)^{\frac{5}{3}}=\int(8x-5)(3x-5)^{\frac{2}{3}}\,dx$

$x(3x-5)^{\frac{5}{3}}=\int 8x(3x-5)^{\frac{2}{3}}-5(3x-5)^{\frac{2}{3}}\,dx$

$x(3x-5)^{\frac{5}{3}}=8\int x(3x-5)^{\frac{2}{3}}\,dx-5\int(3x-5)^{\frac{2}{3}}\,dx$

$x(3x-5)^{\frac{5}{3}}=8\int x(3x-5)^{\frac{2}{3}}\,dx-5\frac{(3x-5)^{\frac{5}{3}}}{(3)\left(\frac{5}{3}\right)}$

$x(3x-5)^{\frac{5}{3}}=8\int x(3x-5)^{\frac{2}{3}}\,dx-(3x-5)^{\frac{5}{3}}$

$8\int x(3x-5)^{\frac{2}{3}}\,dx=x(3x-5)^{\frac{5}{3}}+(3x-5)^{\frac{5}{3}}$

$\int x(3x-5)^{\frac{2}{3}}\,dx=\frac{1}{8}(x+1)(3x-5)^{\frac{5}{3}}+c$

(iii) $\int_{-1}^{\frac{5}{3}}x(3x-5)^{\frac{2}{3}}\,dx$

$=\left[\frac{1}{8}(x+1)(3x-5)^{\frac{5}{3}}\right]_{-1}^{\frac{5}{3}}$

$=\frac{1}{8}\left(\frac{5}{3}+1\right)(0)-\frac{1}{8}(0)(-3-5)=0$

This implies that, for $-1\le x\le\frac{5}{3}$, the total areas between the curve above the x-axis and the x-axis is exactly the same as the total areas between the curve below the x-axis and the x-axis.

Question 8

[Ans: (a) $x=-\ln 2$ (b) $y=\frac{1}{5}$ (c) $e^{\frac{x}{2}}=\frac{2x+7}{3}$; $y=2x+11$]

(a) $e^x\left(1+e^x\right)=\frac{3}{4}$

Let $u=e^x$

$u(1+u)=\frac{3}{4} \Rightarrow u+u^2=\frac{3}{4}$

$4u^2+4u-3=0$

$(2u-1)(2u+3)=0$

$u=\frac{1}{2}$ or $u=-\frac{3}{2}$

$e^x=\frac{1}{2}$ or $e^x=-\frac{3}{2}$ (NA)

$x=\ln\frac{1}{2}=-\ln 2$

(b) $1+\log_2 y+\frac{1}{\log_8 2}=\log_2(y+3)$

$1+\log_2 y+\frac{1}{\frac{\log_2 2}{\log_2 8}}=\log_2(y+3)$

$1+\log_2 y+\log_2 2^3=\log_2(y+3)$

$1+\log_2 y+3=\log_2(y+3)$

$4=\log_2(y+3)-\log_2 y$

$\log_2\left(\frac{y+3}{y}\right)=4$

$\frac{y+3}{y}=2^4$

$y+3=16y$

$15y=3 \Rightarrow y=\frac{1}{5}$

(c) $x=\ln\left\{\left(\frac{2x+7}{3}\right)^2\right\}=2\ln\left(\frac{2x+7}{3}\right)$

$\frac{x}{2}=\ln\left(\frac{2x+7}{3}\right)$

$e^{\frac{x}{2}}=\frac{2x+7}{3}$

$3e^{\frac{x}{2}}=2x+7$

$3e^{\frac{x}{2}}+4=2x+11$

$\therefore$ Equation of the line is $y=2x+11$.

Question 9

[Ans: (i) show (ii) $D(1,8)$; explain (iii) $C(13,13)$]

(i) Gradient of $BD = \frac{6-5}{5-7} = -\frac{1}{2}$

Equation of BD:

$$y-5 = -\frac{1}{2}(x-7) \Rightarrow y = -\frac{1}{2}x + \frac{17}{2} \text{ --- (1)}$$

$$AB = \sqrt{[7-(-5)]^2 + (5-0)^2} = 13$$

Let $D(x, y)$

$$AD = \sqrt{[x-(-5)]^2 + (y-0)^2} = \sqrt{(x+5)^2 + y^2}$$

Perimeter $= 46$

$2AB + 2AD = 46 \Rightarrow AB + AD = 23$

$$13 + \sqrt{(x+5)^2 + y^2} = 23 \Rightarrow \sqrt{(x+5)^2 + y^2} = 10 \Rightarrow (x+5)^2 + y^2 = 100 \text{ --- (2)}$$

Sub. (1) into (2)

$$(x+5)^2 + \left(-\frac{1}{2}x + \frac{17}{2}\right)^2 = 100$$

$$x^2 + 10x + 25 + \frac{1}{4}x^2 - \frac{17}{2}x + \frac{289}{4} = 100$$

$$\frac{5}{4}x^2 + \frac{3}{2}x = \frac{11}{4} \Rightarrow 5x^2 + 6x = 11$$

$\therefore x$-coordinate of D satisfies the equation $5x^2 + 6x = 11$. (shown)

(ii) $5x^2 + 6x = 11$

$5x^2 + 6x - 11 = 0 \Rightarrow (5x+11)(x-1) = 0$

$x = -\frac{11}{5}$ (NA) or $x = 1$

Sub. $x = 1$ into (1) $\quad y = -\frac{1}{2}(1) + \frac{17}{2} = 8$

$\therefore D(1,8)$

The diagram is necessary as it had helped in rejecting $x = -\frac{11}{5}$ as the x-coordinate of D.

(iii) x-coordinate of $C = 1 + [7-(-5)] = 13$

y-coordinate of $C = 8 + (5-0) = 13$

$\therefore C(13,13)$

Question 10

[Ans: (i) $f'(x)=16x^3+e^{2x-1}-3$ (ii) $f(x)=4x^4+\frac{1}{2}e^{2x-1}-3x+1$ (iii) show]

(i) $f''(x)=48x^2+2e^{2x-1}$

$$f'(x)=\int 48x^2+2e^{2x-1}dx$$

$$=48\left(\frac{x^3}{3}\right)+2\left(\frac{e^{2x-1}}{2}\right)+C$$

$$=16x^3+e^{2x-1}+C$$

$$f'\left(\frac{1}{2}\right)=0$$

$$16\left(\frac{1}{2}\right)^3+e^{2\left(\frac{1}{2}\right)-1}+C=0$$

$$2+1+C=0\Rightarrow C=-3$$

$$\therefore f'(x)=16x^3+e^{2x-1}-3$$

(ii) $f(x)$

$$=\int 16x^3+e^{2x-1}-3dx$$

$$=16\left(\frac{x^4}{4}\right)+\left(\frac{e^{2x-1}}{2}\right)-3x+D$$

$$=4x^4+\frac{1}{2}e^{2x-1}-3x+D$$

$$f\left(\frac{1}{2}\right)=\frac{1}{4}$$

$$4\left(\frac{1}{2}\right)^4+\frac{1}{2}e^{2\left(\frac{1}{2}\right)-1}-3\left(\frac{1}{2}\right)+D=\frac{1}{4}$$

$$\frac{1}{4}+\frac{1}{2}-\frac{3}{2}+D=\frac{1}{4}\Rightarrow D=1$$

$$f(x)=4x^4+\frac{1}{2}e^{2x-1}-3x+1$$

(iii) When $x = 0$,

$$y = \mathrm{f}(0)$$
$$= 4(0)^4 + \frac{1}{2}e^{2(0)-1} - 3(0) + 1$$
$$= \frac{1}{2}e^{-1} + 1 = \frac{1}{2e} + 1$$

$$\frac{dy}{dx} = \mathrm{f}'(0)$$
$$= 16(0)^3 + e^{2(0)-1} - 3$$
$$= e^{-1} - 3 = \frac{1}{e} - 3$$

Equation of tangent:

$$y = \left(\frac{1}{e} - 3\right)x + \left(\frac{1}{2e} + 1\right)$$
$$y = \frac{1}{e}x - 3x + \frac{1}{2e} + 1$$

$$y + 3x - 1 = \frac{1}{e}x + \frac{1}{2e}$$
$$2e(y + 3x - 1) = 2x + 1 \text{ (shown)}$$

Question 11

[Ans: (i) show (ii) $\frac{dt}{dx}=-\frac{9\pi}{25k}$ (iii) $k=\frac{3}{50}$]

(i) Let r be the radius of the top surface area of the liquid.

By similar triangles,

$$\frac{r}{x}=\frac{15}{25}\Rightarrow r=\frac{3}{5}x$$

$$V=\frac{1}{3}\pi\left(15^2\right)(25)-\frac{1}{3}\pi r^2(x)$$
$$=\frac{1}{3}\pi\left[5625-\left(\frac{3}{5}x\right)^2 x\right]$$
$$=\frac{1}{3}\pi\left(5625-\frac{9x^3}{25}\right)$$
$$=3\pi\left(625-\frac{x^3}{25}\right) \text{ (shown)}$$

(ii) $\frac{dV}{dx}=3\pi\left(-\frac{3x^2}{25}\right)=-\frac{9}{25}\pi x^2$

$\frac{dV}{dt}=kx^2$ (given)

$$\frac{dV}{dx}=\frac{dV}{dt}\times\frac{dt}{dx}$$
$$-\frac{9}{25}\pi x^2=kx^2\left(\frac{dt}{dx}\right)\Rightarrow\frac{dt}{dx}=-\frac{9\pi}{25k}$$

(iii) $t=\int-\frac{9\pi}{25k}\,dx=-\frac{9\pi}{25k}x+C$

When $t=0$, $x=25$,

$$0=-\frac{9\pi}{25k}(25)+C\Rightarrow C=\frac{9\pi}{k}$$

$$t=-\frac{9\pi}{25k}x+\frac{9\pi}{k}$$

When $t=72\pi$,

$$25-x=12\Rightarrow x=13$$

$$72\pi=-\frac{9\pi}{25k}(13)+\frac{9\pi}{k}$$
$$\frac{108}{25k}=72\Rightarrow k=\frac{3}{50}$$