**RIVER VALLEY HIGH SCHOOL**
**2019 JC2 Preliminary Examination**
Higher 2

| NAME | |
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| CLASS | | | | | |
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| INDEX NUMBER | | |
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# MATHEMATICS 9758/01

Paper 1

**19 September 2019**

**3 hours**

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

**READ THESE INSTRUCTIONS FIRST**

Write your class, index number and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
You are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

| **For examiner's use only** | |
|---|---|
| Question number | Mark |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| 9 | |
| 10 | |
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| | |
| **Total** | |

| **Calculator Model:** |
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This document consists of 5 printed pages.

1 Find the range of values of $x$ that satisfy

$$\frac{2x}{x+5} < \frac{1}{x-1}.$$ [4]

Hence deduce the range of values of $x$ that satisfy

$$\frac{2|x|}{|x|+5} - \frac{1}{|x|-1} < 0.$$ [2]

2. A curve $y = \mathrm{f}(x)$ passes through the point (0 , 2) and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{2y}$.

(i) Obtain the Maclaurin series of $y$ in ascending powers of $x$, up to and including the term in $x^2$.
Write down the equation of the tangent to the curve $y = \mathrm{f}(x)$ at $x = 0$. [4]

(ii) Show that $y = \sqrt{4 + \sin x}$. [2]

(iii) Using your results in part (ii), and assuming $x$ to be sufficiently small for terms in $x^3$ and higher powers to be ignored, obtain the binomial series of $y$. [3]

3. The curve $C$ has equation

$$y = \frac{2x - a}{2x^2 - b},$$

where $a$ and $b$ are positive integers.
$C$ passes through the point $\left(\frac{9}{2}, 0\right)$ and the equation of an asymptote of $C$ is $x = 3$.

(i) Show that $a = 9$ and $b = 18$. [2]

(ii) Sketch $C$, stating clearly the equations of asymptotes, the $x$-coordinates of turning points and axial intercepts. [4]

(iii) Hence, giving your reasons, deduce the range of values of $h$ such that the graph of $\frac{(x-8)^2}{h^2} + \frac{y^2}{100} = 1$, where $h$ is a positive integer, intersects $C$ at exactly 6 distinct points. [2]

4. A function is said to be self-inverse when $\mathrm{f} = \mathrm{f}^{-1}$ for all $x$ in the domain of f.

Given that the function f is defined by

$$\mathrm{f} : x \mapsto \frac{ax+b}{cx-a}, \text{ for } x \in \mathbb{R}, x \neq \frac{a}{c},$$

where $a$, $b$ and $c$ are positive constants.

(i) Show that f is self-inverse. [2]

(ii) Using the result of part (i), deduce $\mathrm{f}^2(x)$ and state the range of $\mathrm{f}^2$. [2]

(iii) Solve the equation $\mathrm{f}^{-1}(x) = x$, leaving your answers in the exact form. [3]

For the rest of the question, let $a = 2$, $b = 5$ and $c = 3$.
The function g is defined by $\mathrm{g} : x \mapsto \mathrm{e}^x + 2$ for $x \in \mathbb{R}$.

(iv) Show that the composite function fg exists, justifying your answer clearly. [1]

(v) By considering the graph of f, or otherwise, find the exact range of fg. [2]

5 Consider the following definitions:

$$\cosh x = \frac{e^{x}+e^{-x}}{2},$$

$$\sinh x = \frac{e^{x}-e^{-x}}{2},$$

$$\tanh x = \frac{\sinh x}{\cosh x},$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \quad \text{and} \quad \operatorname{cosech} x = \frac{1}{\sinh x}.$$

They are known as *hyberbolic functions*. They are used in modeling suspended bridges and equations of motion related to skydiving.

(i) Find an expression for $\tanh x$ in terms of $e^{-2x}$. [1]

(ii) Given that

$$f(n+1) - f(n) = \sinh x\left[\operatorname{sech}\left(n+\frac{1}{2}\right)x\right]\left[\operatorname{sech}\left(n-\frac{1}{2}\right)x\right],$$

where $f(n) = \tanh\left(n-\frac{1}{2}\right)x$ and $x \neq 0$, find an expression for

$$S_N = \sum_{n=1}^{N}\left[\operatorname{sech}\left(n+\frac{1}{2}\right)x\right]\left[\operatorname{sech}\left(n-\frac{1}{2}\right)x\right]$$

in the form $(\operatorname{cosech} x)\left(\tanh Ax - \tanh\frac{1}{2}x\right)$ where $A$ is a constant to be determined. [3]

(iii) Explain why $S_\infty$ exists. Deduce an expression for $S_\infty$ in the form $(\operatorname{cosech} x)(P - \tanh Qx)$, where $P$ and $Q$ are constants to be determined. [3]

6 (a) Find the roots of the equation $(1+i)z^2 - z + (2-2i) = 0$, giving your answers in the form $x+iy$, where $x$ and $y$ are exact real numbers. [4]

(b) Given that $f(z) = z^4 - 2z^3 + z^2 + az + b$, where $a$, $b \in \mathbb{R}$, and that 1 + 2i satisfies the equation f(z) = 0, find the values of $a$ and $b$ and the other roots. [5]

Hence solve the equation $w^4 - 2iw^3 - w^2 - 8iw - 20 = 0$, showing your workings clearly. [2]

7 The area $A$ of the region bounded by $y=\sin^6 x$, the $x$-axis, $x=0$ and $x=\dfrac{\pi}{2}$ is to be found.

(i) The area $A$ can be approximated by dividing the region into 5 vertical strips of rectangles of equal width.
Show that $0.10625\pi < A < 0.20625\pi$. [3]

(ii) It is given that $I_n = \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx$, where $n \geq 0$.
By writing $\sin^{2n} x = (\sin x)(\sin^{2n-1} x)$, show by integration by parts that

$$I_n = \frac{2n-1}{2n} I_{n-1}, \text{ for } n \geq 1.$$

Deduce that $I_2 = \dfrac{3}{16}\pi$.
Using the value of $I_2$, obtain the exact value of $A$. [7]

8. Two planes have equations given by

$$p_1 : x + 5y + 4z = 4,$$
$$p_2 : x - y + z = 4.$$

(i) Explain why $p_1$ and $p_2$ intersect in a line $\ell$ and determine the cartesian equation of $\ell$. [3]

(ii) The plane $p_3$ contains the line $\ell$ and the point $Q(5, 3, -6)$, find a cartesian equation of $p_3$.
Describe the geometrical relationship between these 3 planes and $\ell$. [5]

(iii) The line $m$ passes through the points $S(0,2,0)$ and $T(4,0,0)$. Find the position vector of the foot of perpendicular of $S$ on $p_2$.
Hence find a vector equation of the line of reflection of $m$ in $p_2$. [5]

9 On 1 January 2019, Mr Tan started a savings account with a bank with an initial deposit of \$5000. The bank offered compound interest of 1% computed based on the amount of money in this account at the end of each month. Due to personal financial needs, Mr Tan withdrew \$100 from the savings account at the beginning of each month starting from the second month, i.e. 1 February 2019.

(i) Find the amount of money in Mr Tan's account at the end of March 2019. [3]

(ii) Deduce that the amount of money in Mr Tan's account at the end of the $n^{\text{th}}$ month is given by $\$100\left(101 - 50\left(1.01^n\right)\right)$. [3]

(iii) Determine the month and year Mr Tan will deplete the savings in his account if he continues to withdraw money from this account. [2]

On 1 January 2019, Mrs Tan also started a savings account with the same bank with an initial deposit of \$3000. The bank offered her interest of \$10 for the first month and increment of \$5 for each subsequent month. For example, in the first 3 months, her interest was \$10 for January, \$15 for February and \$20 for March. In addition, Mrs Tan further deposits \$50 into her savings account every mid-month starting from 15 January 2019.

(iv) Determine the month and year Mrs Tan will first have more money in her account than that of Mr Tan by the end of month. [5]

10 Since the 1990s, a group of scientists has started conservation work to prevent the extinction of a rare species of flying fox in the wild Western Australia forests. The number of such species, in thousands, observed in the forests at time $t$ years after the start of the conservation is denoted by $N$. It is known that the death rate of the flying foxes is proportional to the number of flying foxes present and that the birth rate of the flying foxes has been maintained constant at 2000 per year. There were only 1000 flying foxes in the forests at the start of the conservation and the rate of increase of the number of flying foxes was 1500 per year when there were 3000 flying foxes.

(i) Form a differential equation relating $N$ and $t$ and show that it can be reduced to

$$6\frac{\mathrm{d}N}{\mathrm{d}t}=12-N.$$ [2]

(ii) Solve the differential equation to find $N$ in terms of $t$. [5]

(iii) Deduce the minimum number of years in integer, needed for the number of flying foxes to exceed 6000. [2]

(iv) Sketch a graph to show how the number of flying foxes in the forests varies with time and suggest the long term behavior of the flying fox population. [3]

(v) Suggest a possible limitation of the model in part (i). [1]

**END OF PAPER**