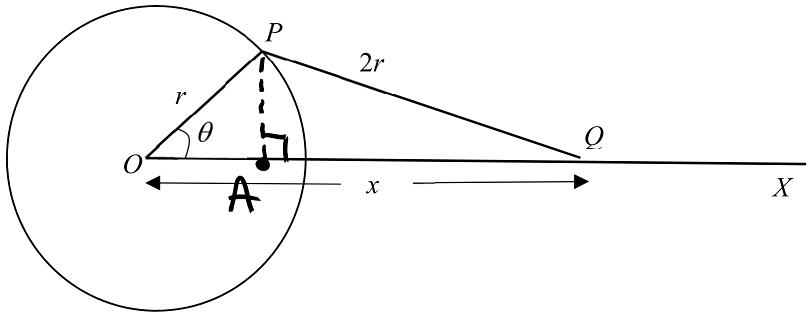
EJC_H2_2019_JC2_Prelim_P2_Solutions

Section A: Pure Mathematics [40 marks]

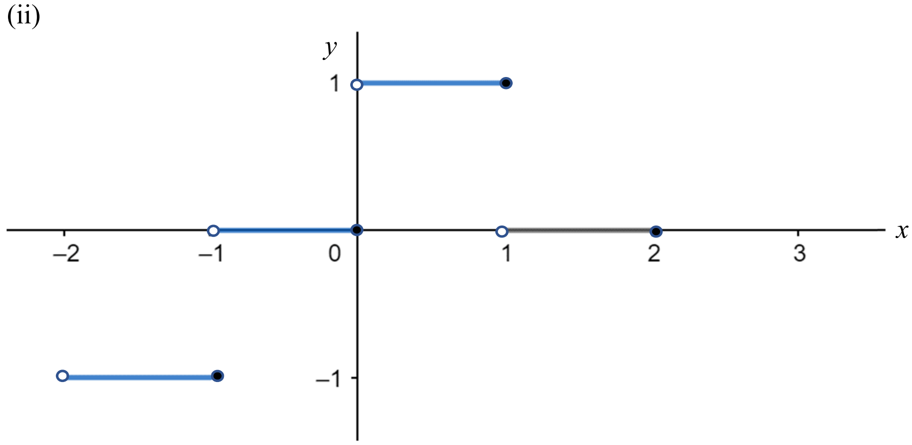
1	(a) Sub $z=(2+2\mathrm{i})w$ into the other equation $\Rightarrow (1-2\mathrm{i})(2+2\mathrm{i})w = 39-11w\mathrm{i}$ $\Rightarrow w = \dfrac{39}{(1-2\mathrm{i})(2+2\mathrm{i})+11\mathrm{i}} = 2-3\mathrm{i}$ (using GC) Thus, $z=(2+2\mathrm{i})(2-3\mathrm{i})=10-2\mathrm{i}$ **Alternative** Sub $w=\dfrac{z}{2+2\mathrm{i}}$ into the other equation
	(b) $(1+\mathrm{i}c)^3 = 1+3\mathrm{i}c+3(\mathrm{i}c)^2+(\mathrm{i}c)^3$ $=1+3\mathrm{i}c-3c^2-\mathrm{i}c^3$ $=1-3c^2+\mathrm{i}(3c-c^3)$ Since $(1+\mathrm{i}c)^3$ is real, $3c-c^3=0$ $c(3-c^2)=0$ $c=0, \pm\sqrt{3}$

2	(i) Max $x=3r$ when $\theta=0$ Min $x=r$ when $\theta=\pi$
	(ii) Method 1 Consider triangle OPA.  $\cos\theta=\dfrac{OA}{r}\Rightarrow OA=r\cos\theta$ Consider triangle PAQ. By pythagoras theorem, $AQ=\sqrt{(2r)^2-(PA)^2}$ $=\sqrt{(2r)^2-(r\sin\theta)^2}$ $=r\sqrt{4-\sin^2\theta}$ $x=OA+AQ=r\cos\theta+r\sqrt{4-\sin^2\theta}=r\left[\cos\theta+\sqrt{4-\sin^2\theta}\right]$ (shown) Method 2: Cosine Rule $(2r)^2=r^2+x^2-2rx\cos\theta$ $4r^2=r^2+x^2-2rx\cos\theta$ $=(x-r\cos\theta)^2+r^2\sin^2\theta$ $x-r\cos\theta=r\sqrt{4-\sin^2\theta}$ (reject $-r\sqrt{4-\sin^2\theta}\because x\geq r\geq r\cos\theta$) $x=r\left(\cos\theta+\sqrt{4-\sin^2\theta}\right)$ (shown)
	(iii) Method 1: $\dfrac{dx}{dt}=\dfrac{dx}{d\theta}\times\dfrac{d\theta}{dt}$ $=r\left[-\sin\theta+\dfrac{(-2\sin\theta\cos\theta)}{2\sqrt{4-\sin^2\theta}}\right]\times\dfrac{d\theta}{dt}$ When $\theta=\dfrac{\pi}{6}$ and $\dfrac{d\theta}{dt}=0.3$, $\dfrac{dx}{dt}=r\left[-\sin\dfrac{\pi}{6}-\dfrac{\sin\frac{\pi}{6}\cos\frac{\pi}{6}}{\sqrt{4-\sin^2\left(\frac{\pi}{6}\right)}}\right](0.3)$ $=-0.217r$

	Method 2: Differentiate implicitly w.r.t t, $\frac{\mathrm{d}x}{\mathrm{d}t}=r\left(\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}+\frac{(-2\sin\theta\cos\theta)}{2\sqrt{4-\sin^2\theta}}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)$ When $\theta=\frac{\pi}{6}$ and $\frac{\mathrm{d}\theta}{\mathrm{d}t}=0.3$, $\frac{\mathrm{d}x}{\mathrm{d}t}=r\left[-\left(\sin\frac{\pi}{6}\right)(0.3)-\frac{\sin\frac{\pi}{6}\cos\frac{\pi}{6}}{\sqrt{4-\sin^2\left(\frac{\pi}{6}\right)}}(0.3)\right]$ $=-0.217r$

3	(i) Length of projection of q onto p $=\lvert\mathbf{q}.\hat{\mathbf{p}}\rvert=\left\lvert\frac{\mathbf{q}.\mathbf{p}}{\lvert\mathbf{p}\rvert}\right\rvert$ Method 1 $3\overrightarrow{PR}=5\overrightarrow{PQ}\Rightarrow 3(\mathbf{r}-\mathbf{p})=5(\mathbf{q}-\mathbf{p})\Rightarrow\mathbf{q}=\frac{1}{5}(2\mathbf{p}+3\mathbf{r})$ Sub into $\lvert\mathbf{q}.\hat{\mathbf{p}}\rvert$: $\lvert\mathbf{q}.\hat{\mathbf{p}}\rvert=\left\lvert\frac{\frac{1}{5}(2\mathbf{p}+3\mathbf{r}).\mathbf{p}}{\lvert\mathbf{p}\rvert}\right\rvert$ $=\left\lvert\frac{\frac{2}{5}\mathbf{p}.\mathbf{p}+\frac{3}{5}\mathbf{p}.\mathbf{r}}{\lvert\mathbf{p}\rvert}\right\rvert$ $=\frac{\frac{2}{5}(29)+\frac{3}{5}(11)}{\sqrt{29}}=\frac{91}{5\sqrt{29}}$ (or 3.38) Method 2 $3\overrightarrow{PR}=5\overrightarrow{PQ}\Rightarrow 3(\mathbf{r}-\mathbf{p})=5(\mathbf{q}-\mathbf{p})\Rightarrow\mathbf{r}=\frac{1}{3}(5\mathbf{q}-2\mathbf{p})$ Sub into $\mathbf{p}.\mathbf{r}=11$: $\Rightarrow\frac{1}{3}(5\mathbf{q}-2\mathbf{p}).\mathbf{p}=11$ $\Rightarrow 5\mathbf{q}.\mathbf{p}-2\mathbf{p}.\mathbf{p}=33$ $\Rightarrow\mathbf{q}.\mathbf{p}=\frac{91}{5}\Rightarrow\lvert\mathbf{q}.\hat{\mathbf{p}}\rvert=\frac{91}{5\sqrt{29}}$ (or 3.38) Method 3 Dot p to both sides, $3(\mathbf{r}-\mathbf{p}).\mathbf{p}=5(\mathbf{q}-\mathbf{p}).\mathbf{p}$

	$\Rightarrow 3(\mathbf{r}-\mathbf{p}).\mathbf{p} = 5(\mathbf{q}-\mathbf{p}).\mathbf{p}$ $\Rightarrow 3\mathbf{r}.\mathbf{p} - 3\mathbf{p}.\mathbf{p} = 5\mathbf{q}.\mathbf{p} - 5\mathbf{p}.\mathbf{p}$ $\Rightarrow \mathbf{p}.\mathbf{q} = \frac{1}{5}(3\mathbf{p}.\mathbf{r} + 2\mathbf{p}.\mathbf{p}) = \frac{1}{5}\left(3(11) + 2\left(\sqrt{29}\right)^2\right) = \frac{91}{5}$ So $\lvert \mathbf{q}.\hat{\mathbf{p}} \rvert = \frac{91}{5\sqrt{29}}$ (or 3.38)
	(ii) $\overrightarrow{PS} = \mathbf{r}$ so *OPSR* is a parallelogram spanned by *OP* and *OR*. So area of *OPSR* $= \lvert \mathbf{p} \times \mathbf{r} \rvert$ $= \left\lvert \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \right\rvert = \left\lvert \begin{pmatrix} -6-8 \\ -(-9+4) \\ -6-2 \end{pmatrix} \right\rvert = \left\lvert \begin{pmatrix} -14 \\ 5 \\ -8 \end{pmatrix} \right\rvert = \sqrt{285}$

4	(i) $\mathrm{f}(-1.4) = \lceil -1.4 \rceil = -1$
	(ii) 
	(iii) Method 1 No, because the horizontal line $y=1$ (for example) cuts the graph more than once from $(0,1]$. So f is not 1-1 so f^{-1} does not exist. Method 2 No, because for example, $\mathrm{f}(1.1) = \mathrm{f}(1.2) = 0$. So f is not 1-1 so f^{-1} does not exist.
	(iv) $R_{\mathrm{f}} = \{-1,0,1\}$
	(v)

	$$g^2(x)=\frac{a\left(\frac{ax-3}{x-a}\right)-3}{\frac{ax-3}{x-a}-a}$$ $$=\frac{a^2x-3a-3x+3a}{ax-3-ax+a^2}$$ $$=x$$ Then $$g^3(x)=g^2(g(x))$$ $$=\frac{ax-3}{x-a}$$ Observe that even compositions give x, odd compositions give g(x). So $g^{2019}(x)=\frac{ax-3}{x-a}\Rightarrow g^{2019}(5)=\frac{5a-3}{5-a}$.
	(vi) $$g(x)=\frac{3x-3}{x-3}$$ $$D_f=(-2,2]\xrightarrow{f}\{-1,0,1\}\xrightarrow{g}\left\{\frac{3}{2},1,0\right\}$$

5	(a)(i) $u_1=\frac{4}{M^2},u_2=\frac{4}{M^5},u_3=\frac{4}{M^8}$
	(a) (ii) $$\sum_{r=1}^{n}\frac{4}{M^{3r-1}}=\frac{\frac{4}{M^2}\left(1-\frac{1}{M^{3n}}\right)}{1-\frac{1}{M^3}}$$ $$=\frac{4}{M^2}\left(1-\frac{1}{M^{3n}}\right)\times\frac{M^3}{M^3-1}$$ $$=\frac{4M}{M^3-1}\left(1-\frac{1}{M^{3n}}\right)$$ (shown)
	(a) (iii) <u>Method 1 (consider expression)</u> Since as $n\to\infty$, $\frac{1}{M^{3n}}\to 0$, $\therefore \frac{4M}{M^3-1}\left(1-\frac{1}{M^{3n}}\right)\to\frac{4M}{M^3-1}$ The sum to infinity is $\frac{4M}{M^3-1}$. <u>Method 2 (consider GP)</u> <u>This is a GP with</u> common ratio = $\frac{1}{M^3}$

	$M>1 \Rightarrow 0<\frac{1}{M}<1 \Rightarrow 0<\frac{1}{M^3}<1$, so the series is convergent. $S=\frac{\frac{4}{M^2}}{1-\frac{1}{M^3}}=\frac{4M}{M^3-1}$
	(b)(i) $\cos\left(\frac{2r+1}{2}\right)-\cos\left(\frac{2r-1}{2}\right)$ $=-2\sin\left(\frac{1}{2}\left(\frac{2r+1}{2}+\frac{2r-1}{2}\right)\right)\sin\left(\frac{1}{2}\left(\frac{2r+1}{2}-\frac{2r-1}{2}\right)\right)$ $=-2\sin(r)\sin\left(\frac{1}{2}\right)$ (shown)
	(b)(ii) $\sum_{r=1}^{n}\sin r=-\frac{1}{2\sin\left(\frac{1}{2}\right)}\sum_{r=1}^{n}\left(\cos\left(\frac{2r+1}{2}\right)-\cos\left(\frac{2r-1}{2}\right)\right)$ $=-\frac{1}{2\sin\left(\frac{1}{2}\right)}\begin{bmatrix}\cos\left(\frac{3}{2}\right)-\cos\left(\frac{1}{2}\right)\\ +\cos\left(\frac{5}{2}\right)-\cos\left(\frac{3}{2}\right)\\ +\cos\left(\frac{7}{2}\right)-\cos\left(\frac{5}{2}\right)\\ + \quad \vdots\\ +\cos\left(\frac{2n-1}{2}\right)-\cos\left(\frac{2n-3}{2}\right)\\ +\cos\left(\frac{2n+1}{2}\right)-\cos\left(\frac{2n-1}{2}\right)\end{bmatrix}$ $=-\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\left[\cos\left(n+\frac{1}{2}\right)-\cos\left(\frac{1}{2}\right)\right]$ $=\operatorname{cosec}\left(\frac{1}{2}\right)\sin\frac{n+1}{2}\sin\frac{n}{2}$ shown

Section B: Probability and Statistics [60 marks]

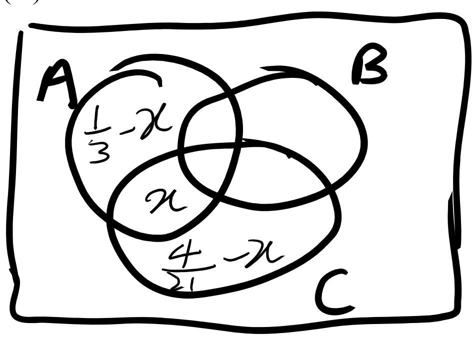
6	(i) $d = 0.5 - c$
	(ii) Let X be the result of one throw of the die. $\mathrm{E}(X) = (1)(0.3) + (2)(c) + (3)(0.5 - c) + (4)(0.2) = 2.6 - c$ $\mathrm{E}(X^2) = (1)(0.3) + (4)(c) + (9)(0.5 - c) + (16)(0.2) = 8 - 5c$ $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ $= (8 - 5c) - (2.6 - c)^2$ $= 8 - 5c - c^2 + 5.2c - 6.76$ $= -c^2 + 0.2c + 1.24$ $= -(c - 0.1)^2 + 1.25$ (completing the square) Thus, the variance is maximum when $c = 0.1$
	(iii) Let Y be the number of throws, out of 10, that land on an even number. $Y \sim \mathrm{B}(10, 0.4)$ Required probability $= \mathrm{P}(Y \geq 7)$ $= 1 - \mathrm{P}(Y \leq 6)$ $= 0.054762...$ $= 0.0548$ (to 3 s.f.)

7	(i) $X_2 \sim \mathrm{N}(\mu, 4)$ $\mathrm{P}(\mu - 1 < X_2 < \mu + 1)$ $= \mathrm{P}\left(\frac{\mu - 1 - \mu}{2} < \frac{X_2 - \mu}{2} < \frac{\mu + 1 - \mu}{2}\right)$ $= \mathrm{P}\left(-\frac{1}{2} < Z < \frac{1}{2}\right)$ where $Z \sim \mathrm{N}(0,1)$ $= 0.38292\ldots$ ≈ 0.383 (3 s.f.)
	(ii) $X_3 - X_4 \sim \mathrm{N}(0, 14)$ $\mathrm{P}(X_3 \geq X_4) = \mathrm{P}(X_3 - X_4 \geq 0) = \frac{1}{2}$ (by symmetry)
	(iii) $\mathrm{Var}(Y_n) = \frac{1}{n^2}\mathrm{Var}(X_1 + X_2 + \ldots + X_n)$ $= \frac{1}{n^2}\left(\mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_n)\right)$ $= \frac{1}{n^2}\left(2 + 4 + 6 + \ldots + 2n\right)$

	$=\frac{1}{n^2}\times n(n+1)$ (Sum of A.P.) $=1+\frac{1}{n}$ Since the X_n's are independent Normal distributions with common mean, $Y_n \sim \mathrm{N}\left(\mu, 1+\frac{1}{n}\right)$ (NB: The variance of Y_n decreases as n increases.) Either $\mathrm{P}(\mu-1<Y_n<\mu+1)>\frac{2}{3}$ $\mathrm{P}\left(\frac{\mu-1-\mu}{\sqrt{1+\frac{1}{n}}}<\frac{Y_n-\mu}{\sqrt{1+\frac{1}{n}}}<\frac{\mu+1-\mu}{\sqrt{1+\frac{1}{n}}}\right)>\frac{2}{3}$ $\mathrm{P}\left(\frac{-1}{\sqrt{1+\frac{1}{n}}}<Z<\frac{1}{\sqrt{1+\frac{1}{n}}}\right)>\frac{2}{3}$ $\frac{1}{\sqrt{1+\frac{1}{n}}}>0.96742$ Solving this inequality, $n>14.6017\ldots$ Hence, the smallest possible value of n is 15. Alternatively $Y_n-\mu \sim \mathrm{N}\left(0, 1+\frac{1}{n}\right)$ From GC, n | $\mathrm{P}(-1<Y_n-\mu<1)$ 14 | 0.6660 15 | 0.6671 $\therefore$ smallest value of n is 15.

n	$\mathrm{P}(-1<Y_n-\mu<1)$
14	0.6660
15	0.6671

8	(i) $\mathrm{P}(B)=\mathrm{P}(A\cup B)-\mathrm{P}(A\cap B')=\frac{6}{7}-\frac{1}{3}=\frac{11}{21}$
	(ii) $\mathrm{P}(A'\mid B)=\frac{\mathrm{P}(A'\cap B)}{\mathrm{P}(B)}=\frac{\mathrm{P}(A\cup B)-\mathrm{P}(A)}{\mathrm{P}(B)}=\frac{\frac{6}{7}-\frac{2}{5}}{\frac{11}{21}}=\frac{\frac{16}{35}}{\frac{11}{21}}=\frac{48}{55}$
	(iii) $\mathrm{P}(B'\cap C)=P(C)-P(B\cap C)$ $=\frac{2}{5}-P(B)P(C)$ ($\because B, C$ independent) $=\frac{2}{5}-\frac{11}{21}\times\frac{2}{5}$ $=\frac{4}{21}$

	(iv)  Let $\mathrm{P}(A \cap B' \cap C) = x$ Since $\frac{4}{21} - x \geq 0$, $x \leq \frac{4}{21}$ Furthermore, since $\mathrm{P}(A \cup B) = \frac{6}{7}$, $\frac{4}{21} - x \leq \frac{1}{7}$, so $x \geq \frac{1}{21}$ <u>Alternative:</u> $A \cap B' \cap C \subseteq B' \cap C \Rightarrow P(A \cap B' \cap C) \leq P(B' \cap C)$ So greatest possible value is $\frac{4}{21}$. Furthermore, $P(A \cap B' \cap C) = P(B' \cap C) - P(A' \cap B' \cap C)$ And $P(A' \cap B' \cap C) \leq P(A' \cap B') = 1 - P(A \cup B) = \frac{1}{7}$ So $P(A \cap B' \cap C) \geq P(B' \cap C) - \frac{1}{7} = \frac{1}{21}$ So least possible value is $\frac{1}{21}$.

9	(a)(i) 9 letters with 3 'E' and 2 'L' No. of ways $= \frac{9!}{3!2!} = 30240$
	(a)(ii) L is fixed _J_W_L_R_Y_: $5! = 120$ Case 1: separated by 2 and 2 – 2 ways Case 2: separated by 2 and 3 / 3 and 2 – 2 ways Total number of ways: $120 \times (2+2) = 480$ ways
	(a)(iii) All distinct: ${}^{6}C_{4} \times 4! = 360$ EE or LL (but not both): ${}^{2}C_{1} \times {}^{5}C_{2} \times \frac{4!}{2!} = 240$ EE and LL: $\frac{4!}{2!2!} = 6$ EEE: ${}^{5}C_{1} \times \frac{4!}{3!} = 20$ Total: $360 + 240 + 6 + 20 = 626$

(b)

Mr and Mrs Lee together: $(9-1)! \times 2! = 80640$

Mr and Mrs Lee together and 3 children together: $(7-1)! \times 2! \times 3! = 8640$

Number of ways: $80640 - 8640 = 72000$

OR

Let A be the event that Mr and Mrs Lee are seated together and
B be the event that the 3 children are all seated together.

Then no. of ways $= n(A) - n(A \cap B)$

$= (9-1)! \times 2! - (7-1)! \times 2! \times 3!$

$= 80640 - 8640 = 72000$

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10 (i)

$\bar{x} = 1050 + \dfrac{58.0}{50} = 1051.16$

$$s^2 = \frac{1}{n-1}\left(\sum (x-1050)^2 - \frac{\left[\sum (x-1050)\right]^2}{n} \right)$$

$$= \frac{1}{49}\left(2326 - \frac{58.0^2}{50} \right)$$

$= 46.096$ (5 s.f.)

To test H$_0$: $\mu = 1053$ against

H$_1$: $\mu \neq 1053$ at 5% level of significance

Since $n = 50$ is large, by Central Limit Theorem,

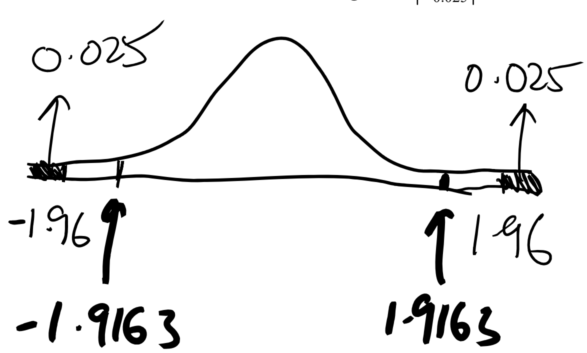
under H$_0$, $\bar{X} \sim \mathrm{N}\left(1053, \dfrac{46.096}{50}\right)$ approximately

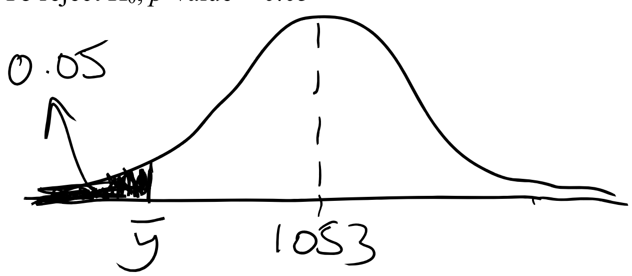
either

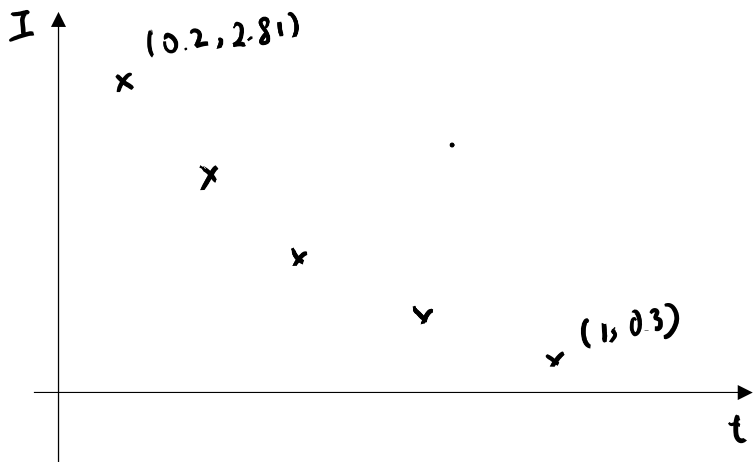
p-value: 0.055322

or

z-value: −1.9163 and critical region: $|z_{0.025}| = 1.96$



	Since p-value > 0.05 (or |z-value| < 1.96), we do not reject H_0 and conclude at 5% level of significance that there is insufficient evidence that the population mean amount of sodium per packet has changed after alterations to the workflow.
	(ii) The probability of wrongly concluding that the mean amount of sodium is not 1053mg, when it is in fact 1053mg, is 0.05.
	(iii) Since p-value is 0.055322, $\alpha \geq 6$
	(iv) If we tested H_1: $\mu < 1053$, Either $p\text{-value} = 0.027661 < 0.05$ or $z\text{-value} = -1.9163 < -1.645$ So we may reject H_0 and conclude at 5% level of significance that the population mean amount of sodium had decreased.
	(v) To test H_0: $\mu = 1053$ against H_1: $\mu < 1053$ at 5% level of significance Since $n = 50$ is large, by Central Limit Theorem, under H_0, $\bar{X} \sim \text{N}\left(1053, \frac{6.0^2}{40}\right)$ approximately To reject H_0, p-value < 0.05  $\Rightarrow \bar{y} < 1051.4$ (to 1 d.p.) It is not necessary to assume anything about the population distribution, as sample size (= 40) is large enough, so the Central Limit Theorem says the sample mean amount of sodium approximately follows a normal distribution.

11	(a) There may be a strong negative linear correlation between the amount of red wine intake and the risk of heart disease, but we cannot conclude that amount of red wine intake causes risk of heart disease to decrease, as causality cannot be inferred from correlation.
	(b)(i) The variable t is the independent variable, as we are able to control, or determine, the intervals at which we measure the corresponding radiation.
	(b)(ii)  From the scatter diagram, we can see that the points lie along a curve, rather than a straight line. Hence $I = at + b$ is not a likely model.
	(b)(iii) r between I and t = -0.9565 r between ln I and t = -0.9998
	(b)(iv) $I = ae^{bt} \Rightarrow \ln I = bt + \ln a$ Equation of regression line: $\ln I = -2.7834239t + 1.6007544 \Rightarrow \ln I = -2.78t + 1.60$ $\ln a = 1.600754 \Rightarrow a = 4.96$ (3 s.f.) $b = -2.78$ (3 s.f.)
	(b)(v) $t = 0.7$, $I = 0.706$ (to 3 sig fig) The answer is reliable as r is close to -1, and $t = 0.7$ is within the data range (0.2 to 1.0) and thus the estimate is obtained via interpolation.

End of Paper