2019 Year 6 H2 Math Prelim P2 Mark Scheme

Qn	Suggested Solution	
1(i)	$S_n - S_{n-1}$ $= an^2 + bn + c - (a(n-1)^2 + b(n-1) + c)$ $= 2an - a + b$ Total number of additional cards need is $2an - a + b$	
(ii)	Additional cards to form 2nd level from 1st level $= 5$ $4a - a + b = 5 \Rightarrow 3a + b = 5$ --- (1) Additional cards to form 3rd level from 2nd level $= 8$ $6a - a + b = 8 \Rightarrow 5a + b = 8$ ---(2) Solving both (1) and (2), $a = \frac{3}{2}, b = \frac{1}{2}$. Using $S_1 = 2 \Rightarrow \frac{3}{2}(1)^2 + \frac{1}{2}(1) + c = 2 \Rightarrow c = 0.$ **Alternative** Substituting different values of *n*, $n = 1$: $a + b + c = 2$ $n = 2$: $4a + 2b + c = 7$ $n = 3$: $9a + 3b + c = 15$ From GC, $a = 1.5$, $b = 0.5$ and $c = 0$ **Alternative** $n = 1$, number of cards = 2 $n = 2$, number of cards = 2 + 5 $n = 3$, number of cards = 2 + 5 + 8 $S_n = \frac{n}{2}[2(2) + (n-1)(3)] = \frac{n}{2}(3n+1) = 1.5n^2 + 0.5n$ $\therefore$ $a = 1.5$, $b = 0.5$ and $c = 0$	
(ii)	$u_n = 3n - 1$ $u_n - u_{n-1} = (3n-1) - (3(n-1)-1) = 3$ (constant) Thus Sn is a sum of AP with common difference 3.	
(iii)	$\sum_{n=1}^{23} S_n = \sum_{n=1}^{23} (1.5n^2 + 0.5n) = 6624$	

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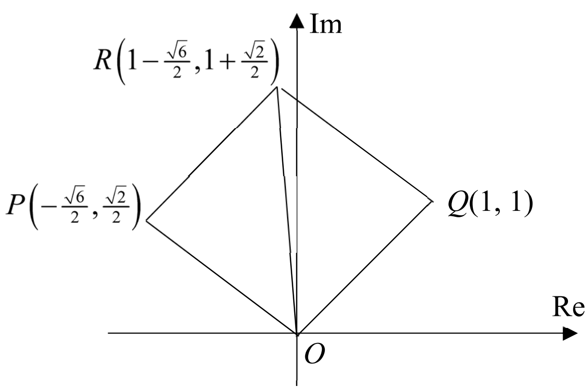
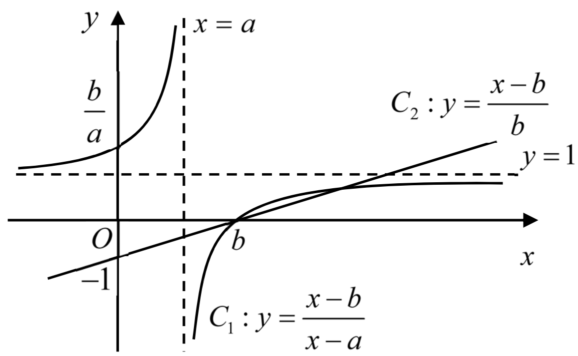
2 (i)	$3x^2-2xy+5y^2=14$ ---- (1) Differentiate (1) implicitly wrt x: $6x-2x\frac{\mathrm{d}y}{\mathrm{d}x}-2y+10y\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $(2x-10y)\frac{\mathrm{d}y}{\mathrm{d}x}=6x-2y$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3x-y}{x-5y}$ (shown)	
(ii)	$x-5y=0 \Rightarrow y=0.2x$ Sub $y=0.2x$ into (1): $3x^2-2x(0.2x)+5(0.2x)^2=14$ $2.8x^2=14$ $x=\pm\sqrt{5}$	
(iii)	When $y=1$, $3x^2-2x-9=0$ Therefore, $x=-1.4305$ or $x=2.0972$ $\frac{\mathrm{d}y}{\mathrm{d}t}=(\frac{\mathrm{d}y}{\mathrm{d}x})(\frac{\mathrm{d}x}{\mathrm{d}t})$ $-7=(\frac{3x-1}{x-5})(\frac{\mathrm{d}x}{\mathrm{d}t})$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{7(5-x)}{3x-1}$ When $x=2.0972$, $\frac{\mathrm{d}x}{\mathrm{d}t}=3.84$ units per second (3 s.f.)	

Qn	Suggested Solution	
3(i)	LHS $=a\left(\frac{1}{z_0}\right)^2+b\left(\frac{1}{z_0}\right)+a$ $=\left(\frac{1}{z_0}\right)^2\left(a+bz_0+az_0{}^2\right)$ $=0$ $\quad\because a+bz_0+az_0{}^2=0$ Thus $z=\frac{1}{z_0}$ is a solution. Since a and b are real constants,	

	$\frac{1}{z_0} = z_0^*$ $z_0 z_0^* = 1$ $\lvert z_0 \rvert^2 = 1$ Since $\lvert z_0 \rvert > 0$, $\lvert z_0 \rvert = 1$ **Alternative for first part:** Let second root be z_1 product of roots $z_0 z_1 = \frac{a}{a} = 1$ $\therefore z_1 = \frac{1}{z_0}$	
(ii)	Let $z_0 = x_0 + \mathrm{i}y_0$ Since $\mathrm{Im}(z_0) = \frac{1}{2}$, $y_0 = \frac{1}{2}$. From part (i), $\lvert z_0 \rvert = 1$ $\sqrt{x_0^2 + y_0^2} = 1$ $\sqrt{x_0^2 + \left(\frac{1}{2}\right)^2} = 1$ $x_0 = \pm\frac{\sqrt{3}}{2}$ $z_0 = \frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}$ or $-\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}$	
(iii)	Since $\mathrm{Re}(z_0) > 0$, $z_0 = \frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}$. Subst into $az_0^2 + bz_0 + a = 0$, $a\left(\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}\right)^2 + b\left(\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}\right) + a = 0$ $a\left(\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right) + b\left(\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}\right) + a = 0$ $\left(\frac{3}{2}a + \frac{\sqrt{3}}{2}b\right) + \mathrm{i}\left(\frac{1}{2}b + \frac{\sqrt{3}}{2}a\right) = 0$ $\therefore b = -\sqrt{3}a$	

Qn	Suggested Solution	
4(i)	$w=\sqrt{2}\left(\cos\frac{1}{4}\pi+\mathrm{i}\sin\frac{1}{4}\pi\right)$ $=1+\mathrm{i}$ $z=\sqrt{2}\left(\cos\frac{5}{6}\pi+\mathrm{i}\sin\frac{5}{6}\pi\right)$ $=-\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}$ $w+z=\left(1-\frac{\sqrt{6}}{2}\right)+\left(1+\frac{\sqrt{2}}{2}\right)\mathrm{i}$	
(ii)	 *OPRQ* is a rhombus	
(iii)	Note that *OR* bisects the angle *POQ* since *OPRQ* is a rhombus. Thus $\arg(w+z)=\frac{1}{2}\left(\frac{1}{4}\pi+\frac{5}{6}\pi\right)=\frac{13}{24}\pi.$ $\tan\left(\frac{11}{24}\pi\right)=\frac{1+\frac{\sqrt{2}}{2}}{\frac{\sqrt{6}}{2}-1}$ $=\frac{2+\sqrt{2}}{\sqrt{6}-2}$ $\therefore a=2, b=-2$	

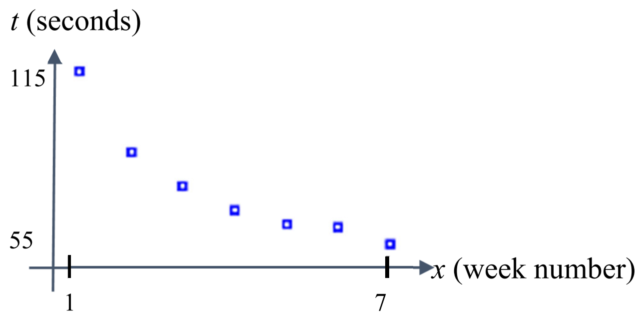
Qn	Suggested Solution	
5(i)	Graphs intersect at:	

	$\frac{x-b}{x-a}=\frac{x-b}{b}$ $b(x-b)=(x-b)(x-a)$ $(x-b)(x-a-b)=0$ $x=b$ or $x=a+b$  	
(ii)	$\therefore x<a$ or $b\le x\le a+b$	
(iii)	From GC, point of intersection at $(5,\frac{2}{3})$ $V=\pi\int_0^{\frac{2}{3}} \underbrace{x_2^{\ 2}}_{C_2}-\underbrace{x_1^{\ 2}}_{C_1}\ \mathrm{d}y$ $=\pi\int_0^{\frac{2}{3}}(3y+3)^2-\left(\frac{2y-3}{y-1}\right)^2\ \mathrm{d}y$ $=5.742$ (3 d.p.)	

Qn	Suggested Solution	
6	For distinct gifts, 5^6 ways Now considering the distinct gifts, Case 1: 3 person get 1 gift No of ways = ${}^5C_3\times5^6=156250$ Case 2: 1 person get 1 gift, another person gets 2 gifts No of ways = ${}^5C_2\left(2\right)\times5^6=312500$ Case 3: 1 person get 3 gifts No of ways = ${}^5C_1\times5^6=78125$ Total number of ways $=156250+312500+78125=546875$ **Alternative**	

	<u>Stage 1: Distribute 6 distinct gifts among 5 people</u> No of ways = 5^6 <u>Stage 2: Distribute 3 identical gifts among 5 people</u> Case 1: 3 person get 1 gift No of ways = ${}^5C_3$ = 10 Case 2: 1 person get 1 gift, another person gets 2 gifts No of ways = ${}^5C_2\left(2\right)$ = 20 Case 3: 1 person get 3 gifts No of ways = ${}^5C_1$= 5 Total number of ways = (10+20+5)5^6 = 546875	

Qn	Suggested Solution (updated 26 Sep)	
7(i)	$\mathrm{P}(L'\cup M')=\frac{80-n(L\cap M)}{80}$ (4 to 6 hours) $=\frac{80-(35-k)}{80}=\frac{45+k}{80}$ **ALT** $\mathrm{P}(L'\cup M')=\mathrm{P}(L)+P(M')-\mathrm{P}(L'\cap M')$ $=\frac{10+k}{80}+\frac{35}{80}-0$ $=\frac{45+k}{80}$	
(ii)	$\mathrm{P}(G\mid L')=\frac{P(G\cap L')}{P(L')}=\frac{k}{k+10}$	
(iii)	Given $\mathrm{P}(L\cap M)=\frac{2}{5}$ From table: $\mathrm{P}(L\cap M)=\frac{20+(15-k)}{80}=\frac{35-k}{80}$ Solving: $k=3$ $P(L)P(M)=\frac{67}{80}\times\frac{45}{80}=\frac{603}{1280}\neq\frac{2}{5}$ Since $P(L\cap M)\neq P(L)P(M)$, L and M are **NOT** independent **ALT** $P(L)=\frac{70-k}{80}=\frac{67}{80}$ $\mathrm{P}(L\mid M)=\frac{35-k}{45}=\frac{32}{45}\neq\frac{67}{80}$ Since $P(L)\neq\mathrm{P}(L\mid M)$, L and M are **NOT** independent	
(iv)	Since $\mathrm{P}(G\cap(L\cap M))=0$ $\Rightarrow 15-k=0$ $\therefore k=15$	

Qn	Suggested Solution	
8 (i)		
(ii)	A linear model would predict her timing to decrease at a constant rate and eventually negative, which is not possible as there is a limit to how fast a person can swim. A quadratic model would predict that her timings would have a minimum and then increase at an increasing rate, which is also not appropriate.	
(iii)	Based on the scatter diagram and the model, as x increases t decreases at a decreasing rate, therefore b is positive. a has to be positive as it represents the best possible timing that Sharron can swim in the long run.	
(iv)	From GC, $r = 0.991$ $b = 67.69$ $a = 49.50$	
(v)	Let m be the best timing Sharron has at the 8th month. $\overline{\left(\frac{1}{x}\right)} = 0.33973$ We know that $\left(\overline{\frac{1}{x}}, \bar{t}\right)$ is on the regression line $t = 48.28 + 69.45\left(\frac{1}{x}\right)$. $\bar{t} = 48.28 + 69.45(0.33973) = 71.874$ $\frac{522+m}{8} = 71.874$ $m = 52.992$ Sharron best timing is 53 seconds at the 8th month	

Qn	Suggested Solution	
9 (a)	An unbiased estimate for the population variance : $s^2 = \frac{n}{n-1}\left(4^2\right) = \frac{16n}{n-1}$ minutes^2	
(b) (i)	Let μ be the population mean time taken for a 17-year-old student to complete a 5 km run. To test at 10 % significance level, $H_0 : \mu = 30.0$ min $H_1 : \mu \neq 30.0$ min For $n = 40$, $s^2 = \frac{16(40)}{39} = \frac{640}{39}$ Test Statistic: Under H_0 , $\overline{T} \sim \mathrm{N}\left(30.0, \frac{640/39}{40}\right)$ approximately by Central Limit Theorem since n is large p-value $= 2\mathrm{P}\left(\overline{T} \leq 28.9\right) = 0.0859 \leq 0.10$, we reject H_0 and conclude that there is sufficient evidence at the 10 % significance level that the population mean time taken has changed.	
(ii)	The p-value is the probability of obtaining a sample mean at least as extreme as the given sample, assuming that the population mean time taken has not changed from 30.0 min. **OR** The p-value is the smallest significance level to conclude that the population mean time has changed from 30.0 min.	
(iii)	Since the sample size of 40 is large, by Central Limit Theorem, $\overline{T}$ follows a normal distribution approximately. Thus no assumptions are needed.	
(c) (i)	New population mean timing = $0.95 \times 30 = 28.5$ min To test at 5 % significance level, $H_0 : \mu = 28.5$ min $H_1 : \mu > 28.5$ min	
(ii)	Assumption: n is large for Central Limit Theorem to apply. Test Statistic: Under H_0, $\overline{T} \sim \mathrm{N}\left(28.5, \frac{4.0^2}{n-1}\right)$ approximately by Central Limit Theorem	

	For H_0 to be rejected, we need $\mathrm{P}\left(\overline{T} \geq 28.9\right) \leq 0.01$ $\mathrm{P}\left(Z \geq \frac{28.9-28.5}{\frac{4}{\sqrt{n-1}}}\right) \leq 0.01$ $\mathrm{P}\left(Z \geq \frac{\sqrt{n-1}}{10}\right) \leq 0.01$ $\frac{\sqrt{n-1}}{10} \geq 2.3263$ $n \geq 542.2$ Thus required set = $\left\{n \in \mathbb{Z} : n \geq 543\right\}$	

Qn	Suggested Solution	
10 (i)	By symmetry, $\mu = \dfrac{5.2+7.0}{2} = 6.1$ $\text{P}(Y < 5.2) = \text{P}(Y \geq 7.0) = 0.379$ $\text{P}\left(Z < \dfrac{5.2-6.1}{\sigma}\right) = 0.379 \Rightarrow \dfrac{-0.9}{\sigma} = -0.308108$ $\sigma = 2.92105 = 2.92$ (3sf)	
(ii)	$X \sim N(12.3, 9.9)$ $\text{P}(\lvert X-12.3\rvert < a) = 0.5$ $\text{P}(12.3-a < X < 12.3+a) = 0.5$ From GC, $12.3-a = 10.1777$ $a = 2.1223 = 2.12$ (3sf)	
	Alternative $\text{P}(\lvert X-12.3\rvert < a) = 0.5$ $\text{P}(\lvert Z\rvert < \dfrac{a}{\sqrt{9.9}}) = 0.5$ $\text{P}(Z < -\dfrac{a}{\sqrt{9.9}}) = 0.25 \Rightarrow -\dfrac{a}{\sqrt{9.9}} = -0.674489$ $a = 2.12$ (3sf)	
(iii)	$\text{P}(X > 10 = 0.76761$ Let W = number of e-scooters that exceed speed limit, out of 49 $W \sim \text{B}(49, \text{P}(X > 10))$ i.e. $W \sim \text{B}(49, 0.76761)$ Probability required $= \text{P}(W = 34) \times 0.76761$ $= 0.61022 \times 0.76761$ $= 0.046840 = 0.0468$ (3sf)	
(iv)	Want: $\text{P}\left(\dfrac{X_1+\ldots+X_6}{6} > 2\left(\dfrac{Y_1+\ldots+Y_{15}}{15}\right)\right)$ $= \text{P}(\bar{X} - 2\bar{Y} > 0)$ $\bar{X} - 2\bar{Y} \sim \text{N}\left(12.3-2(6.1),\ \dfrac{9.9}{6} + \dfrac{4}{15}\left(2.92105^2\right)\right)$ i.e. $\bar{X} - 2\bar{Y} \sim \text{N}(0.1,\ 3.92533)$ $\therefore \text{P}(\bar{X} - 2\bar{Y} > 0) = 0.520$ (3sf)	

(v)	Let T = Total speed of n e-scooters $\overline{T} \sim \text{N}(12.3, \frac{9.9}{n})$ $\text{P}(\overline{T} > 10) = \text{P}(Z > \frac{10-12.3}{\sqrt{\frac{9.9}{n}}})$ $= \text{P}(Z > -0.73098\sqrt{n}) = 1$ (since n is large) **<u>Alternative</u>** As n gets larger, $\overline{x} \to \mu = 12.3 > 10$ Thus mean speed of these n e-scooters > 10 with probability 1	

Qn	Suggested Solution	
11 (a)(i)	**Method 1: direct computation** $\text{P}(2 \le X \le k)$ $= \text{P}(X=2)+\text{P}(X=3)+\text{P}(X=4)+...+\text{P}(X=k)$ $= (\tfrac{5}{6})(\tfrac{1}{6})+(\tfrac{5}{6})^2(\tfrac{1}{6})+(\tfrac{5}{6})^3(\tfrac{1}{6})+(\tfrac{5}{6})^4(\tfrac{1}{6})+...+(\tfrac{5}{6})^{k-1}(\tfrac{1}{6})$ $= (\tfrac{1}{6})\left[(\tfrac{5}{6})+(\tfrac{5}{6})^2+(\tfrac{5}{6})^3+(\tfrac{5}{6})^4+...+(\tfrac{5}{6})^{k-1}\right]$ $= (\tfrac{1}{6})\left[\dfrac{(\tfrac{5}{6})(1-(\tfrac{5}{6})^{k-1})}{1-(\tfrac{5}{6})}\right]$ $= (\tfrac{5}{6})-(\tfrac{5}{6})^k$ **Method 2: complement method** $\text{P}(2 \le X \le k)$ $= 1-\text{P}(X=1)-\underbrace{\text{P}(X>k)}_{\text{first } k \text{ are not } 6'\text{s}}$ $= 1-\tfrac{1}{6}-(\tfrac{5}{6})^k$ $= \tfrac{5}{6}-(\tfrac{5}{6})^k$ <table><tr><td>s</td><td>8</td><td>4</td><td>0</td></tr><tr><td>$\text{P}(S=s)$</td><td>$\tfrac{1}{6}$</td><td>$(\tfrac{5}{6})-(\tfrac{5}{6})^k$</td><td>$(\tfrac{5}{6})^k$</td></tr></table>	
(ii)	From GC, $\text{E}(S)=\dfrac{8}{6}+4\left(\dfrac{5}{6}-\left(\dfrac{5}{6}\right)^k\right)=\dfrac{14}{3}-4\left(\dfrac{5}{6}\right)^k$ $\text{E}(\text{Profit})=\dfrac{14}{3}-4\left(\dfrac{5}{6}\right)^k-3>0$ $\dfrac{14}{3}-4\left(\dfrac{5}{6}\right)^k-3>0$ $\left(\dfrac{5}{6}\right)^k<\dfrac{5}{12}$ $k>4.802$ Least value of k is 5.	
(b)(i)	$Y \sim \text{B}(80, p)$ $80+80p=480p(1-p)$ $1+p=6p-6p^2$ $6p^2-5p+1=0$ $p=\dfrac{1}{3}$ or $p=\dfrac{1}{2}$ (rejected as coin is not fair)	

(ii)	Let W be the number of heads obtained in the last 75 tosses $W \sim \text{B}(75, \frac{1}{3})$ Required probability $= \text{P}(W \geq 25)$ $= 1 - \text{P}(W \leq 24)$ $= 0.543$ **Alternative** Use conditional probability	
(iii)	$\overline{Y} \sim \text{N}(\frac{80}{3}, \frac{16}{45})$ approximately by central limit theorem since the sample size of 50 is large $\text{P}(\overline{Y} < 25) = 0.00259$ (3 s.f.)	