ANDERSON SERANGOON JUNIOR COLLEGE

MATHEMATICS **9758**

H2 Mathematics Paper 1 (100 marks)

30 August 2019

3 hours

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS | | | / | | |

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

1 **(a)** Show that $\sqrt{\dfrac{1-2x}{1+2x}}$ can be written in the form $\dfrac{\mathrm{f}(x)}{\sqrt{1-4x^2}}$ where f(x) is a polynomial to be determined. Hence, find $\displaystyle\int\sqrt{\frac{1-2x}{1+2x}}\,\mathrm{d}x$ [3]

<u>Solution</u>

$$\int\sqrt{\frac{1-2x}{1+2x}}\,\mathrm{d}x = \int\frac{\sqrt{1-2x}}{\sqrt{1+2x}}\times\frac{\sqrt{1-2x}}{\sqrt{1-2x}}\,\mathrm{d}x$$

$$= \int\frac{1-2x}{\sqrt{1-4x^2}}\,\mathrm{d}x$$

$$= \int\frac{1}{\sqrt{1-4x^2}}-\frac{2x}{\sqrt{1-4x^2}}\,\mathrm{d}x$$

$$= \frac{1}{2}\int\frac{2}{\sqrt{1-(2x)^2}}\,\mathrm{d}x+\frac{1}{2}\int\left(\frac{-8x}{2}\right)\left(1-4x^2\right)^{\frac{-1}{2}}\,\mathrm{d}x$$

$$= \frac{1}{2}\sin^{-1}2x+\frac{1}{2}\sqrt{1-4x^2}+c$$, where c is an arbitrary constant.

(b) Show that $\displaystyle\int_{\mathrm{e}}^{\mathrm{e}^2}\frac{1}{x\ln x}\,\mathrm{d}x = \ln 2$. [3]

<u>Solution</u>

$$\int_{\mathrm{e}}^{\mathrm{e}^2}\frac{1}{x\ln x}\,\mathrm{d}x = \int_{\mathrm{e}}^{\mathrm{e}^2}\frac{\frac{1}{x}}{\ln x}\,\mathrm{d}x$$

$$= \left[\ln\left|\ln x\right|\right]_{\mathrm{e}}^{\mathrm{e}^2}$$

$= \ln|\ln \mathrm{e}^2| - \ln|\ln \mathrm{e}|$

$= \ln 2 - \ln 1$

$= \ln 2$ (Shown)

2 It is given that $\tan\frac{1}{2}y = \sqrt{2}x$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

(i) Show that $\left(1+2x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{2}$. [2]

Solution

$$\tan\frac{1}{2}y = \sqrt{2}x \qquad \text{---(1)}$$

Differentiate w.r.t x, $\frac{1}{2}\sec^2\left(\frac{1}{2}y\right)\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{2}$

$$\left[1+\tan^2\left(\frac{1}{2}y\right)\right]\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{2}$$

$$\left(1+2x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{2} \qquad \text{---(2)}$$

(ii) Use the result from part **(i)** to find the first two non-zero terms in the Maclaurin series for y, giving the coefficients in exact form. [3]

Solution

Differentiate Eq(2) w.r.t x, $\left(1+2x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \qquad \text{---(3)}$

$$\left(1+2x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \qquad \text{---(4)}$$

When $x = 0$, from (1), (2), (3) & (4),

$y = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{2}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -8\sqrt{2}$

$$\therefore \quad y = 0 + 2\sqrt{2}x + 0x^2 + \frac{-8\sqrt{2}}{3!}x^3 + \ldots$$

$$= 2\sqrt{2}x - \frac{4\sqrt{2}}{3}x^3 + \ldots$$

(iii) Hence, using standards series from the List of Formulae (MF26), find the expansion of $\dfrac{2\tan^{-1}\sqrt{2}x}{\cos 2x}$ in ascending powers of x, up to and including the term in x^3, giving the coefficients in exact form. [3]

<u>Solution</u>

$$\frac{2\tan^{-1}\sqrt{2}x}{\cos 2x} = \frac{y}{\cos 2x}$$

$$= \left(2\sqrt{2}x - \frac{4\sqrt{2}}{3}x^3 + \ldots\right)\left[1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 + \ldots\right]^{-1}$$

$$= \left(2\sqrt{2}x - \frac{4\sqrt{2}}{3}x^3 + \ldots\right)\left[1 - 2x^2 + \frac{2}{3}x^4 + \ldots\right]^{-1}$$

$$= \left(2\sqrt{2}x - \frac{4\sqrt{2}}{3}x^3 + \ldots\right)\left[1 - \left(-2x^2 + \frac{2}{3}x^4 + \ldots\right)\right]$$

$$= \left(2\sqrt{2}x - \frac{4\sqrt{2}}{3}x^3 + \ldots\right)\left(1 + 2x^2 + \ldots\right)$$

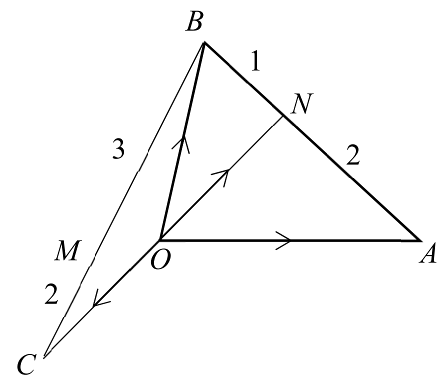
$$= 2\sqrt{2}x + 4\sqrt{2}x^3 - \frac{4\sqrt{2}}{3}x^3 + \ldots$$

$$= 2\sqrt{2}x + \frac{8\sqrt{2}}{3}x^3 \text{ (up to the } x^3 \text{ term)}$$

3 The position vectors of A, B and C referred to a point O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. The point N is on AB such that AN:NB = 2:1.

(i) If O is the midpoint of CN, prove that $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c} = \mathbf{0}$. [2]

<u>Solution</u>



By ratio theorem, $\overrightarrow{ON} = \frac{1}{3}(\mathbf{a} + 2\mathbf{b})$

Since O is the midpoint of CN, $\overrightarrow{ON} = -\mathbf{c}$

$\therefore\ \overrightarrow{ON} = \frac{1}{3}(\mathbf{a} + 2\mathbf{b}) = -\mathbf{c}$

$\Rightarrow \mathbf{a} + 2\mathbf{b} = -3\mathbf{c}$

$\Rightarrow \mathbf{a} + 2\mathbf{b} + 3\mathbf{c} = \mathbf{0}$

The point M is on CB such that CM:MB = 2:3.

(ii) Show that A, O and M are collinear, and find the ratio AO:OM [3]

<u>Solution</u>

By the ratio theorem, $\overrightarrow{OM} = \frac{2\mathbf{b} + 3\mathbf{c}}{2+3}$

$$= \frac{-\mathbf{a}}{5} = -\frac{1}{5}\overrightarrow{OA}$$

Hence, A, O and M are collinear (Shown)

Since $\left|\overrightarrow{OM}\right| = \frac{1}{5}\left|\overrightarrow{OA}\right|$, the ratio AO:OM = 5:1

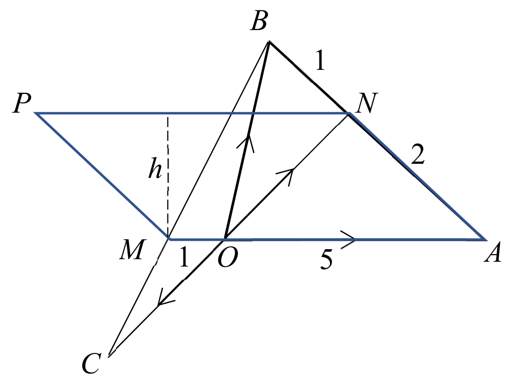
(iii) If the point P is such that $\overrightarrow{NP} = \overrightarrow{AM}$, show that the ratio of the area of $PNAM$: area of $PNOM$ = 12:7 [3]

Solution

Method 1

Area of //gram $PNAM$

$= h\left|\overrightarrow{AM}\right|$ where h is the height of the //gram



Area of trapezium $PNOM$

$= \frac{1}{2}\left(\left|\overrightarrow{PN}\right|+\left|\overrightarrow{OM}\right|\right)h$

$= \frac{1}{2}\left(\left|\overrightarrow{AM}\right|+\frac{1}{6}\left|\overrightarrow{AM}\right|\right)h$

$= \frac{7}{12}h\left|\overrightarrow{AM}\right| = \frac{7}{12} \times$ Area of //gram $PNAM$

$\therefore$ the ratio of the area of $PNAM$: area of $PNOM$ = 12:7

Method 2

Area of //gram $PNAM$

$=\left|\overrightarrow{PN}\times\overrightarrow{PM}\right|$

$=\left|\frac{6}{5}\underset{\sim}{a}\times\left(\frac{2}{3}\underset{\sim}{a}-\frac{2}{3}\underset{\sim}{b}\right)\right| = \frac{4}{5}\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|$

Area of trapezium $PNOM$

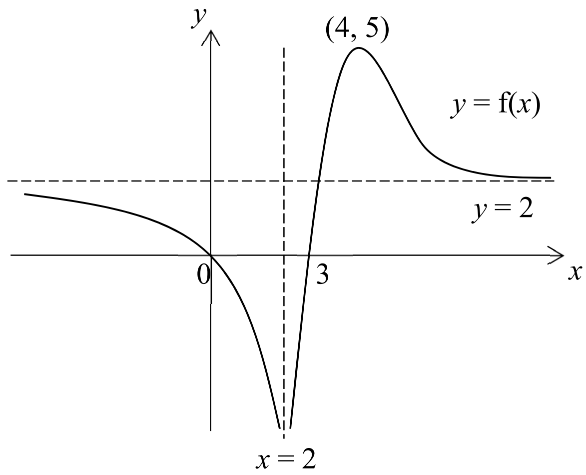
$=\left|\overrightarrow{PN}\times\overrightarrow{PM}\right|-\frac{1}{2}\left|\overrightarrow{ON}\times\overrightarrow{OA}\right|$

$=\frac{4}{5}\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|-\frac{1}{2}\left|\left(\frac{1}{3}\underset{\sim}{a}+\frac{2}{3}\underset{\sim}{b}\right)\times\underset{\sim}{a}\right|$

$=\frac{4}{5}\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|-\frac{1}{2}\left|-\frac{2}{3}\underset{\sim}{a}\times\underset{\sim}{b}\right| = \frac{7}{15}\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|$

$\therefore$ the ratio of the area of $PNAM$: area of $PNOM$ = 12:7

4 **(a)**

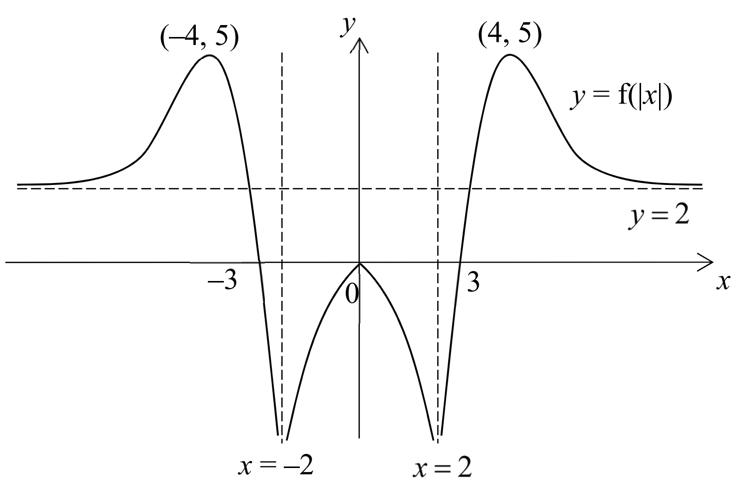


The diagram shows the graph of $y = \mathrm{f}(x)$. The curve crosses the x-axis at $x = 0$ and $x = 3$. It has a turning point at (4, 5) and asymptotes with equations $y = 2$ and $x = 2$.

Showing clearly the coordinates of turning points, axial intercepts and equations of asymptotes where possible, sketch the graphs of

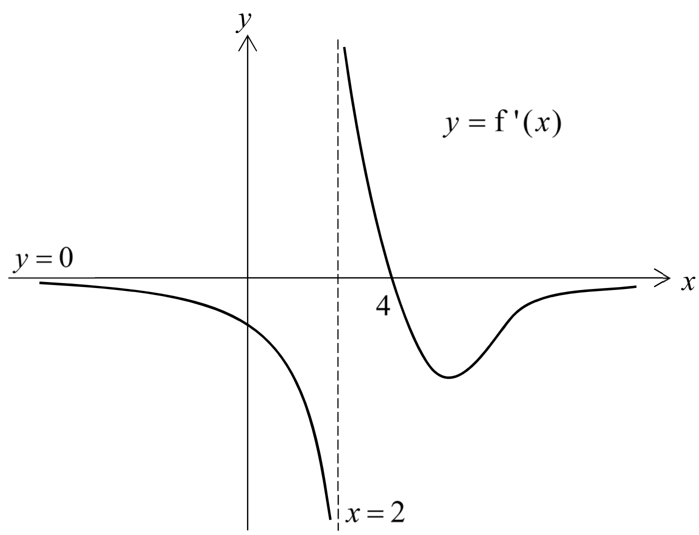
(i) $y = \mathrm{f}\left(|x|\right)$; [2]

<u>Solution</u>



(ii) $y = \mathrm{f}'(x)$ [3]

Solution



(**b**) The curve with equation $y = \mathrm{f}(x)$ is transformed by a stretch with scale factor 2 parallel to the x–axis, followed by a translation of 2 units in the negative x–direction, followed by a translation of 3 units in the positive y–direction. The equation of the resulting curve is $y = \ln\left(\mathrm{e}^3 x\right)$.

Find the equation of the curve $y = \mathrm{f}(x)$. [3]

Solution

Translation of 3 units in the negative y-direction

$$y = \ln\left(\mathrm{e}^3 x\right) \xrightarrow{\text{replace } y \text{ with } y+3} y = \ln\left(\mathrm{e}^3 x\right) - 3 = 3 + \ln x - 3 = \ln x$$

Translation of 2 units in the positive x-direction

$$y = \ln x \xrightarrow{\text{replace } x \text{ with } x-2} y = \ln(x-2)$$

Stretch with scale factor $\frac{1}{2}$ parallel to the $x-$axis

$$y = \ln(x-2) \xrightarrow{\text{replace } x \text{ with } 2x} y = \ln(2x-2)$$

5 A curve C has parametric equations

$$x = a\sin^2 t, \qquad y = a\cos t,$$

where $0 \le t \le \dfrac{\pi}{2}$ and $a > 0$.

(i) Find the cartesian equation of C, stating clearly any restrictions on the values of x and y. [2]

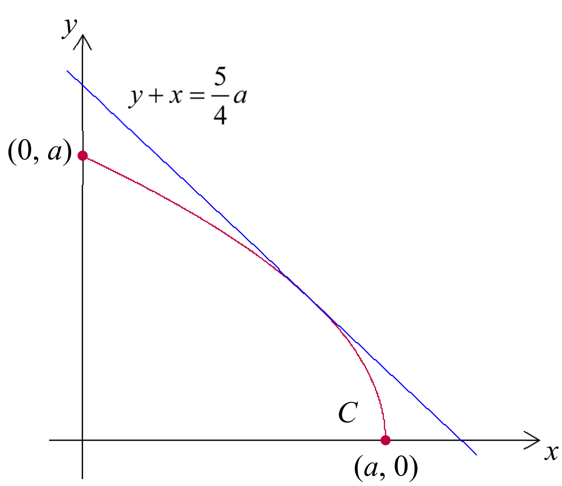
Solution

$\sin^2 t + \cos^2 t = 1 \quad \Rightarrow \quad \dfrac{x}{a} + \left(\dfrac{y}{a}\right)^2 = 1$

$ax + y^2 = a^2$ where $0 \le x \le a$ and $0 \le y \le a$

(ii) Sketch C, showing clearly the axial intercepts. [1]

Solution



(iii) The region bounded by C, the line $y+x=\frac{5}{4}a$ and the y-axis is rotated through 2π radians about the y-axis. Show that the exact volume of the solid obtained is $k\pi a^3$ where k is a constant to be determined. [5]

Solution

$$y+x=\frac{5}{4}a \Rightarrow x=\frac{5}{4}a-y$$

$$\therefore a\left(\frac{5}{4}a-y\right)+y^2=a^2$$

$$y^2-ay+\frac{1}{4}a^2=0$$

$$\left(y-\frac{1}{2}a\right)^2=0$$

$$\therefore y=\frac{1}{2}a$$

$\Rightarrow$ Volume required

$$=\frac{1}{3}\pi\left(\frac{5}{4}a-\frac{1}{2}a\right)^2\left(\frac{5}{4}a-\frac{1}{2}a\right)-\pi\int_{\frac{1}{2}a}^{a}\left(\frac{a^2-y^2}{a}\right)^2\,\mathrm{d}y$$

$$=\frac{1}{3}\pi\left(\frac{3}{4}a\right)^3-\frac{\pi}{a^2}\int_{\frac{1}{2}a}^{a}a^4-2a^2y^2+y^4\,\mathrm{d}y$$

$$=\frac{9}{64}\pi a^3-\frac{\pi}{a^2}\left[a^4y-\frac{2}{3}a^2y^3+\frac{1}{5}y^5\right]_{\frac{1}{2}a}^{a}$$

$$=\frac{9}{64}\pi a^3-\frac{\pi}{a^2}\left[a^5-\frac{2}{3}a^5+\frac{1}{5}a^5-\frac{1}{2}a^5+\frac{1}{12}a^5-\frac{1}{160}a^5\right]$$

$$=\frac{29}{960}\pi a^3\text{ units}^3\ ;\ k=\frac{29}{960}\qquad\text{(Shown)}$$

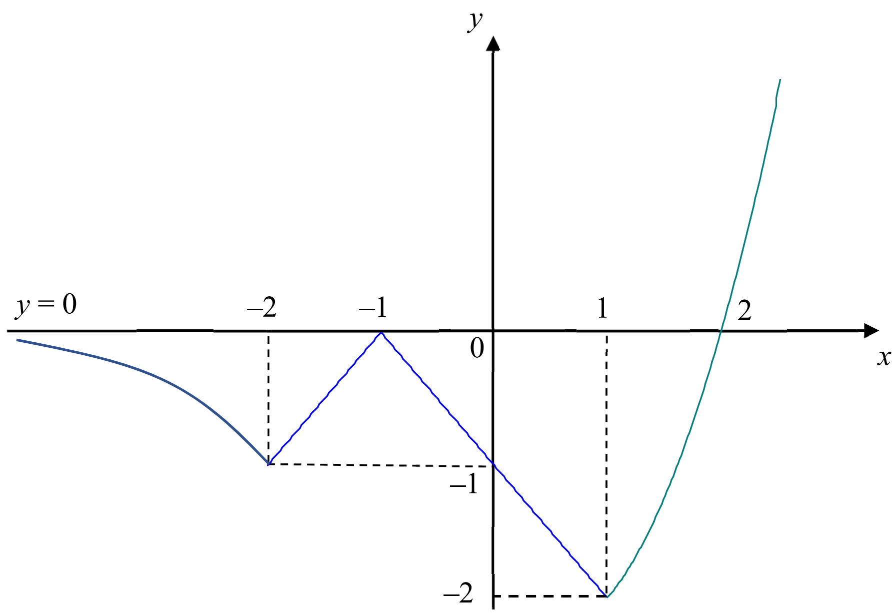
6 A function f is defined by

$$\mathrm{f}(x)=\begin{cases}\dfrac{6}{x-4} & \text{for } x<-2,\\ -|1+x| & \text{for } -2\le x\le 1,\\ x^2-x-2 & \text{for } x>1.\end{cases}$$

(i) Sketch the graph of f. [3]

Solution

(i)



(ii) Evaluate exactly $\int_{-3}^{4}|\mathrm{f}(x)|\,\mathrm{d}x$. [4]

$$\int_{-3}^{4}|\mathrm{f}(x)|\,\mathrm{d}x=\int_{-3}^{-2}\frac{-6}{x-4}\,\mathrm{d}x+\frac{1}{2}(1)(1)+\frac{1}{2}(2)(2)-\int_{1}^{2}x^2-x-2\,\mathrm{d}x+\int_{2}^{4}x^2-x-2\,\mathrm{d}x$$

$$=-6\left[\ln|x-4|\right]_{-3}^{-2}+\frac{5}{2}-\left[\frac{1}{3}x^3-\frac{1}{2}x^2-2x\right]_{1}^{2}+\left[\frac{1}{3}x^3-\frac{1}{2}x^2-2x\right]_{2}^{4}$$

$$=-6\ln\frac{6}{7}+\frac{5}{2}-\left[\frac{8}{3}-\frac{4}{2}-4-\frac{1}{3}+\frac{1}{2}+2\right]+\left[\frac{64}{3}-\frac{16}{2}-8-\frac{8}{3}+\frac{4}{2}+4\right]$$

$$=-6\ln\frac{6}{7}+\frac{37}{3}$$

7 Do not use a calculator in answering this question.

The roots of the equation $z^2 + (2 - 2i)z = -3 - 2i$ are z_1 and z_2.

(i) Find z_1 and z_2 in cartesian form $x + iy$, showing clearly your working. [5]

Solution

(i) Let $z = x + iy$

$\therefore (x + iy)^2 + (2 - 2i)(x + iy) = -3 - 2i$

$x^2 + 2xyi - y^2 + 2x + 2yi - 2xi + 2y = -3 - 2i$

$(x^2 + 2x - y^2 + 2y) + i(2xy + 2y - 2x) = -3 - 2i$

By comparing real and imaginary parts,

$x^2 + 2x - y^2 + 2y = -3$ …(1)

$2xy + 2y - 2x = -2$

$y(x + 1) = x - 1$

$y = \dfrac{x-1}{x+1}$ …(2)

Subt (2) into (1):

$$x^2 + 2x - \left(\frac{x-1}{x+1}\right)^2 + 2\left(\frac{x-1}{x+1}\right) = -3$$

$$x^2(x+1)^2 + 2x(x+1)^2 - (x-1)^2 + 2(x-1)(x+1) = -3(x+1)^2$$

$$x^2(x^2+2x+1) + 2x(x^2+2x+1) - (x^2-2x+1) + 2(x^2-1) = -3(x^2+2x+1)$$

$x^4 + 4x^3 + 9x^2 + 10x = 0$

Let $f(x) = x(x^3 + 4x^2 + 9x + 10)$

$f(0) = 0$ and $f(-2) = -8 + 16 - 18 + 10 = 0$

$\Rightarrow x = 0, -2$ are roots of $x^4 + 4x^3 + 9x^2 + 10x = 0$

When $x = 0$, $y = -1$ and when $x = -2$, $y = 3$

$\therefore z_1 = -i$ and $z_2 = -2 + 3i$

(ii) The complex numbers z_1 and z_2 are also roots of the equation

$$z^4 + 4z^3 + 14z^2 + 4z + 13 = 0 .$$

Find the other roots of the equation, explaining clearly how the answers are obtained. [2]

Solution

Since all the coefficients of $z^4 + 4z^3 + 14z^2 + 4z + 13 = 0$ are real, the other roots of the equation must be complex conjugates of $-\mathrm{i}$ and $-2 + 3\mathrm{i}$.

Hence the other two roots are i and $-2 - 3\mathrm{i}$.

(iii) Using your answer in part **(i)**, solve $z^2 + (2 + 2\mathrm{i})z = 3 + 2\mathrm{i}$. [2]

Solution

$z^2 + (2 - 2\mathrm{i})z = -3 - 2\mathrm{i}$

$-z^2 + (-2 + 2\mathrm{i})z = 3 + 2\mathrm{i}$

$-z^2 + (2\mathrm{i}^2 + 2\mathrm{i})z = 3 + 2\mathrm{i}$

$(\mathrm{i}z)^2 + (2\mathrm{i} + 2)\mathrm{i}z = 3 + 2\mathrm{i}$

Hence from part (i), $-\mathrm{i}z = -\mathrm{i}$ or $-2 + 3\mathrm{i}$

$$\Rightarrow\ z = 1 \text{ or } \frac{-2+3\mathrm{i}}{-\mathrm{i}} \times \frac{\mathrm{i}}{\mathrm{i}} = -3 - 2\mathrm{i}$$

8 **(i)** Show that $\dfrac{3r+4}{(r+2)(r+1)r}=\dfrac{A}{r+2}+\dfrac{B}{r+1}+\dfrac{C}{r}$ where A, B and C are constants to be determined. [1]

Solution

$$\frac{3r+4}{(r+2)(r+1)r}=\frac{A}{r+2}+\frac{B}{r+1}+\frac{C}{r}$$

By cover-up rule, $A=\dfrac{-2}{(-1)(-2)}=-1, B=\dfrac{1}{(1)(-1)}=-1, C=\dfrac{4}{(2)(1)}=2$

(ii) Find the sum to n terms of

$$\frac{7}{3\times2\times1}+\frac{10}{4\times3\times2}+\frac{13}{5\times4\times3}+...$$

(There is no need to express your answer as a single algebraic fraction). [4]

Solution

The sum to nth terms $=\displaystyle\sum_{r=1}^{n}\frac{3r+4}{(r+2)(r+1)r}$

$$=\sum_{r=1}^{n}\left[-\frac{1}{r+2}-\frac{1}{r+1}+\frac{2}{r}\right]$$

$$=\left[-\frac{1}{3}-\frac{1}{2}+2\right.$$
$$-\frac{1}{4}-\frac{1}{3}+1$$
$$-\frac{1}{5}-\frac{1}{4}+\frac{2}{3}$$
$$+...$$
$$-\frac{1}{n+1}-\frac{1}{n}+\frac{2}{n-1}$$
$$\left.-\frac{1}{n+2}-\frac{1}{n+1}+\frac{2}{n}\right]$$

$$=\frac{5}{2}-\frac{1}{n+2}-\frac{2}{n+1}$$

(iii) Hence show that $\displaystyle\sum_{r=5}^{n+3} \frac{3r-5}{(r-1)(r-2)(r-3)} < \frac{4}{3}$. [3]

Solution

$$\sum_{r=5}^{n+3} \frac{3r-5}{(r-1)(r-2)(r-3)}$$

$$= \sum_{r+3=5}^{r+3=n+3} \frac{3(r+3)-5}{(r+3-1)(r+3-2)(r+3-3)}$$

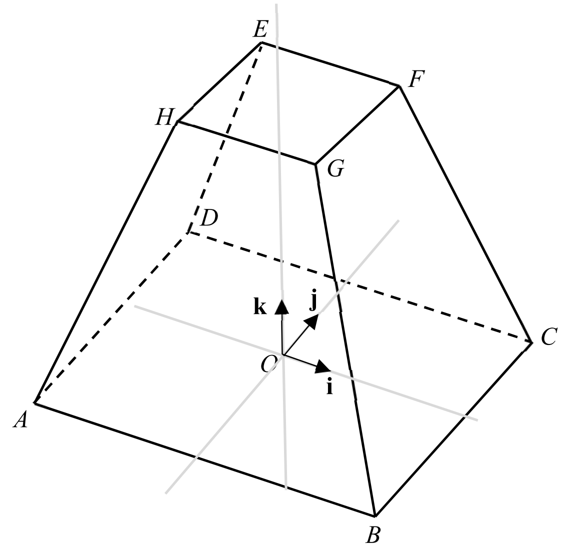
$$= \sum_{r=2}^{r=n} \frac{3r+4}{(r+2)(r+1)(r)}$$

$$= \sum_{r=1}^{n} \frac{3r+4}{(r+2)(r+1)(r)} - \frac{3(1)+4}{(3)(2)(1)}$$

$$= \left[\frac{5}{2} - \frac{1}{n+2} - \frac{2}{n+1}\right] - \frac{7}{6}$$

$$= \frac{4}{3} - \left[\frac{1}{n+2} + \frac{2}{n+1}\right] < \frac{4}{3} \text{ since } \frac{1}{n+2} + \frac{2}{n+1} > 0 \text{ for all } n.$$

9



The diagram above shows an object with O at the centre of its rectangular base $ABCD$ where AB = 8 cm and BC = 4 cm. The top side of the object, $EFGH$ is a square with side 2 cm long and is parallel to the base. The centre of the top side is vertically above O at a height of h cm.

(i) Show that the equation of the line BG may be expressed as $\mathbf{r}=\begin{pmatrix}4\\-2\\0\end{pmatrix}+t\begin{pmatrix}-3\\1\\h\end{pmatrix}$,

where t is a parameter. [1]

<u>Solution</u>

$B(4,-2,0)$ and $G(1,-1,h)$

$$\overrightarrow{BG}=\begin{pmatrix}1\\-1\\h\end{pmatrix}-\begin{pmatrix}4\\-2\\0\end{pmatrix}=\begin{pmatrix}-3\\1\\h\end{pmatrix}$$

$$l_{BG}:\underset{\sim}{r}=\begin{pmatrix}4\\-2\\0\end{pmatrix}+t\begin{pmatrix}-3\\1\\h\end{pmatrix}, t\in\mathbb{R}$$

(ii) Find the sine of the angle between the line BG and the rectangular base $ABCD$ in terms of h. [2]

Solution

Let θ be the angle between the line BG and the rectangular base $ABCD$.

$$\sin\theta = \frac{1}{\sqrt{10+h^2}}\begin{pmatrix}-3\\1\\h\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\sin\theta = \frac{h}{\sqrt{10+h^2}}$$

It is given that $h = 6$.

(iii) Find the cartesian equation of the plane $BCFG$. [3]

Solution

A normal perpendicular to plane $BCFG$ is

$$\underset{\sim}{n} = \overrightarrow{BG}\times\overrightarrow{GF} = \begin{pmatrix}-3\\1\\6\end{pmatrix}\times\begin{pmatrix}0\\2\\0\end{pmatrix} = \begin{pmatrix}-12\\0\\-6\end{pmatrix} = -6\begin{pmatrix}2\\0\\1\end{pmatrix}$$

Equation of plane $BCFG$ is $\underset{\sim}{r}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix}\cdot\begin{pmatrix}4\\-2\\0\end{pmatrix}$

$$\underset{\sim}{r}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix} = 8$$

Cartesian equation of plane is $2x+z-8=0$

(iv) Find the shortest distance from the point A to the plane $BCFG$. [2]

Solution

Shortest distance from point A to plane $BCFG = \dfrac{\left|\underset{\sim}{a}\cdot\underset{\sim}{n}-d\right|}{\left|\underset{\sim}{n}\right|}$

$$= \frac{1}{\sqrt{5}}\left|\begin{pmatrix}-4\\-2\\0\end{pmatrix}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix}-8\right| = \frac{16}{\sqrt{5}} = \frac{16\sqrt{5}}{5} \text{ units}$$

(v) The line l, which passes through the point A, is parallel to the normal of plane $BCFG$. Given that, the line l intersects the plane $BCFG$ at a point M, use your answer in part (iv) to find the shortest distance from point M to the rectangular base $ABCD$. [2]

Solution

$$\overrightarrow{AM} = \frac{16}{\sqrt{5}}\hat{\underset{\sim}{n}} = \frac{16}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}\begin{pmatrix}2\\0\\1\end{pmatrix}\right) = \frac{16}{5}\begin{pmatrix}2\\0\\1\end{pmatrix}$$

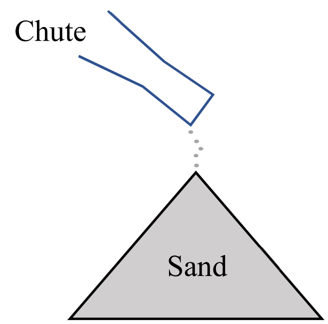
Shortest distance from M to the base = z-coordinate of $\overrightarrow{AM}$

$$= \frac{16}{5}$$

10 Commonly used in building materials, sand is the second largest world resource used by humans after water. To reduce the environmental impacts of sand mining, an alternative approach is to make "sand" by crushing rock.

A machine designed for this purpose produces sand in large quantities. Sand falling from the output chute of the machine forms a pile in the shape of a right circular cone such that the height of the cone is always equal to $\frac{4}{3}$ of the radius of its base.

[Volume of a cone = $\frac{1}{3}\pi r^2 h$ and curved surface area of a cone = $\pi r l$ where r is the radius of the base area, h is the height of the cone and l is the slant length of the cone]



A machine operator starts the machine.

(a) **(i)** Given that V and A denote the volume and the curved surface area of the conical pile respectively, write down V and A in terms of r, the radius of its base. [2]

<u>Solution</u>

$$V = \frac{1}{3}\pi r^2\left(\frac{4}{3}r\right) = \frac{4}{9}\pi r^3$$

$$A = \pi r l$$

$$= \pi r\sqrt{r^2 + h^2}$$

$$= \pi r\sqrt{r^2 + \frac{16}{9}r^2} = \frac{5}{3}\pi r^2$$

(ii) Hence show that the rate of change of A with respect to V is inversely proportional to the radius of the conical pile. [3]

<u>Solution</u>

$$\frac{dV}{dr} = \frac{4}{9}\pi\left(3r^2\right) = \frac{4}{3}\pi r^2 \qquad \text{and} \qquad \frac{dA}{dr} = \frac{5}{3}\pi\left(2r\right) = \frac{10}{3}\pi r$$

$$\frac{dA}{dV}=\frac{dA}{dr}\times\frac{dr}{dV}$$

$$=\frac{10}{3}\pi r\times\frac{1}{\frac{4}{3}\pi r^2}$$

$$=\frac{5}{2r}\qquad\text{(Shown)}$$

An architect is tasked to design sand-lined walking paths in a large park. He decides to base his design of the paths on the shape of astroids, which are shapes with equations

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=k^{\frac{2}{3}}\ (k>0).$$

On a piece of graph paper, he sketches an astroid with the equation $x^{\frac{2}{3}}+y^{\frac{2}{3}}=k^{\frac{2}{3}}$.

(b) The tangent at a point $P(x_1, y_1)$ on the curve meets the x-axis at Q and the y-axis at R. Show that the length of QR is independent of where P lies on the curve. [7]

Solution

(ii) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=k^{\frac{2}{3}}$

Differentiating wrt x, $\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0$

$$\frac{dy}{dx}=-\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

$\therefore$ the equation of the tangent at $P(x_1, y_1)$ is $y-y_1=\left(-\frac{y_1}{x_1}\right)^{\frac{1}{3}}(x-x_1)$

When $x=0$, $y-y_1=\left(-\frac{y_1}{x_1}\right)^{\frac{1}{3}}(-x_1)$

$$y=y_1^{\frac{1}{3}}x_1^{\frac{2}{3}}+y_1$$

When $y = 0$, $\quad -y_1 = \left(-\dfrac{y_1}{x_1}\right)^{\frac{1}{3}}\left(x - x_1\right)$

$$x = y_1^{\frac{2}{3}} x_1^{\frac{1}{3}} + x_1$$

The length of QR $= \sqrt{x^2 + y^2}$

$$= \sqrt{\left(y_1^{\frac{2}{3}} x_1^{\frac{1}{3}} + x_1\right)^2 + \left(y_1^{\frac{1}{3}} x_1^{\frac{2}{3}} + y_1\right)^2}$$

$$= \sqrt{x_1^{\frac{2}{3}}\left(y_1^{\frac{2}{3}} + x_1^{\frac{2}{3}}\right)^2 + y_1^{\frac{2}{3}}\left(x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}}\right)^2}$$

$$= \sqrt{x_1^{\frac{2}{3}}\left(k^{\frac{2}{3}}\right)^2 + y_1^{\frac{2}{3}}\left(k^{\frac{2}{3}}\right)^2}$$

$$= k^{\frac{2}{3}}\sqrt{x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}}}$$

$$= k^{\frac{2}{3}}\sqrt{k^{\frac{2}{3}}}$$

$= k$ which is a constant.

Hence, the length of QR is independent of where P lies on the curve. (Shown)

11 **(a)** Find the sum of all integers between 200 and 1000 (both inclusive) that are not divisible by 7. [4]

Solution

Sum required $= (200 + 201 + 202 + \ldots + 1000) - (203 + 210 + 217 + \ldots + 994)$

$= \frac{1000-200+1}{2}(200+1000) - 7(29 + 30 + 31 + \ldots + 142)$

$= 480600 - 7\left(\frac{142-29+1}{2}\right)(29+142)$

$= 412371$

(b)

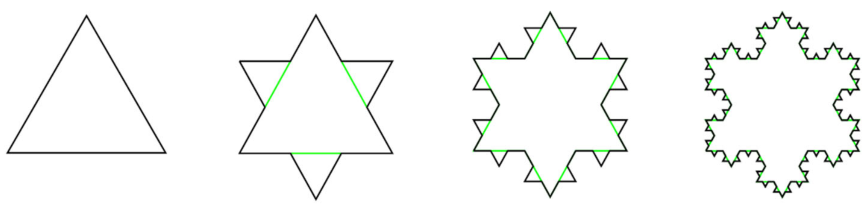


Fig. 1 **Fig. 2** **Fig. 3** **Fig. 4**

Snowflakes can be constructed by starting with an equilateral triangle (Fig. 1), then repeatedly altering each line segment of the resulting polygon as follows:

1. Divide each outer line segments into three segments of equal length.
2. Add an equilateral triangle that has the middle segment from step 1 as its base.
3. Remove the line segment that is the base of the triangle from step 2.
4. Repeat the above steps for a number of iterations, *n*.

The 1st, 2nd and 3rd iteration produces the snowflakes in Fig. 2, Fig. 3 and Fig. 4 respectively.

(i) If a_0 denotes the area of the original triangle and the area of each new triangle added in the n^{th} iteration is denoted by a_n, show that

$$a_n = \frac{1}{9}a_{n-1} \text{ for all positive integers } n.$$ [2]

Solution

Each new triangle added in the n^{th} iteration is similar to the triangle added in the previous iteration (AAA).

Since the length of a side of the new triangle is $\frac{1}{3}$ of the length of a side of the triangle in the previous iteration, $a_n = \left(\frac{1}{3}\right)^2 a_{n-1} = \frac{1}{9}a_{n-1}$ (Shown)

(ii) Write down the number of sides in the polygons in Fig.1, Fig. 2 and Fig. 3, and deduce, with clear explanations, that the number of new triangles added in the n^{th} iteration is $T_n = k\left(4^n\right)$ where k is a constant to be determined. [2]

Solution

No of sides in the polygon (Fig 1) = 3, No of sides in the polygon (Fig 2) = 12

No of sides in the polygon (Fig 3) = 48

GP with first term 3 and common ratio 4

Since there is 1 new triangle for every side in the next iteration, the number of new triangles added in the n^{th} iteration = the number of sides in the polygon after the $(n-1)^{\text{th}}$ iteration

i.e. $T_1 = 3$, $T_2 = 12$, $T_3 = 48$, …

$\therefore T_n = 3(4)^{n-1} = \frac{3}{4}(4)^n$ and $k = \frac{3}{4}$

(iii) Find the total area of triangles added in the n^{th} iteration, A_n in terms of a_0 and n. [2]

Solution

$$A_n = T_n \times a_n = \frac{3}{4}\left(4^n\right)\times\left(\frac{1}{9}a_{n-1}\right)$$

$$= \frac{3}{4}\left(4^n\right)\times\left(\underbrace{\frac{1}{9}\times\frac{1}{9}\times\frac{1}{9}\times\ldots\times\frac{1}{9}}_{n\text{ terms}}a_0\right) = \frac{3}{4}\left(\frac{4}{9}\right)^n a_0 \text{ units}^2$$

(iv) Show that the total area of the snowflake produced after the n^{th} iteration is $a_0\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right]$ units^2. [3]

Solution

$$\text{Total area required} = a_0 + \sum_{r=1}^{n}\frac{3}{4}\left(\frac{4}{9}\right)^r a_0$$

$$= a_0 + \frac{3}{4}a_0\frac{\frac{4}{9}\left(1-\frac{4}{9}^n\right)}{1-\frac{4}{9}}$$

$$= a_0\left[1+\frac{3}{5}\left(1-\frac{4}{9}^n\right)\right]$$

$$= a_0\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right] \qquad \text{(Shown)}$$

The snowflake formed when the above steps are followed indefinitely is called the Koch snowflake.

(v) Determine the least number of iterations needed for the area of the snowflake to exceed 99% of the area of a Koch snowflake. [3]

Solution

$$\text{Area of a Koch snowflake} = \lim_{n\to\infty} a_0\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right] = \frac{8}{5}a_0$$

$$\text{Consider } a_0\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right] > 0.99\times\frac{8}{5}a_0$$

$$0.016-\frac{3}{5}\left(\frac{4}{9}\right)^n > 0$$

From GC,

n	$0.016-\frac{3}{5}\left(\frac{4}{9}\right)^n$
4	$-0.007 < 0$
5	$0.0056 > 0$
6	$0.0114 > 0$

Hence the least value of n is 5.