A-LEVEL H2 MATHS 2019 – PAPER 2

Question 1

[Ans: (i) $-\frac{2}{3}x(1-x)^{\frac{3}{2}}-\frac{4}{15}(1-x)^{\frac{5}{2}}+A$ (ii) $-\frac{2}{3}(1-x)^{\frac{3}{2}}+\frac{2}{5}(1-x)^{\frac{5}{2}}+B$ (iii) show]

(i) Let $u=x$, $\frac{dv}{dx}=(1-x)^{\frac{1}{2}}$

$$\frac{du}{dx}=1,\ v=\frac{(1-x)^{\frac{3}{2}}}{(-1)\left(\frac{3}{2}\right)}=-\frac{2}{3}(1-x)^{\frac{3}{2}}$$

$$\begin{aligned}
I&=\int x(1-x)^{\frac{1}{2}}\,dx\\
&=-\frac{2}{3}x(1-x)^{\frac{3}{2}}-\int -\frac{2}{3}(1-x)^{\frac{3}{2}}\,dx\\
&=-\frac{2}{3}x(1-x)^{\frac{3}{2}}+\frac{2}{3}\frac{(1-x)^{\frac{5}{2}}}{(-1)\left(\frac{5}{2}\right)}+A\\
&=-\frac{2}{3}x(1-x)^{\frac{3}{2}}-\frac{4}{15}(1-x)^{\frac{5}{2}}+A
\end{aligned}$$

(ii) $u^2=1-x \Rightarrow x=1-u^2 \Rightarrow \frac{dx}{du}=-2u$

$$\begin{aligned}
I&=\int x(1-x)^{\frac{1}{2}}\,dx\\
&=\int (1-u^2)(u^2)^{\frac{1}{2}}(-2u)\,du\\
&=-2\int u^2-u^4\,du\\
&=-2\left(\frac{u^3}{3}-\frac{u^5}{5}\right)+B\\
&=-\frac{2}{3}(1-x)^{\frac{3}{2}}+\frac{2}{5}(1-x)^{\frac{5}{2}}+B
\end{aligned}$$

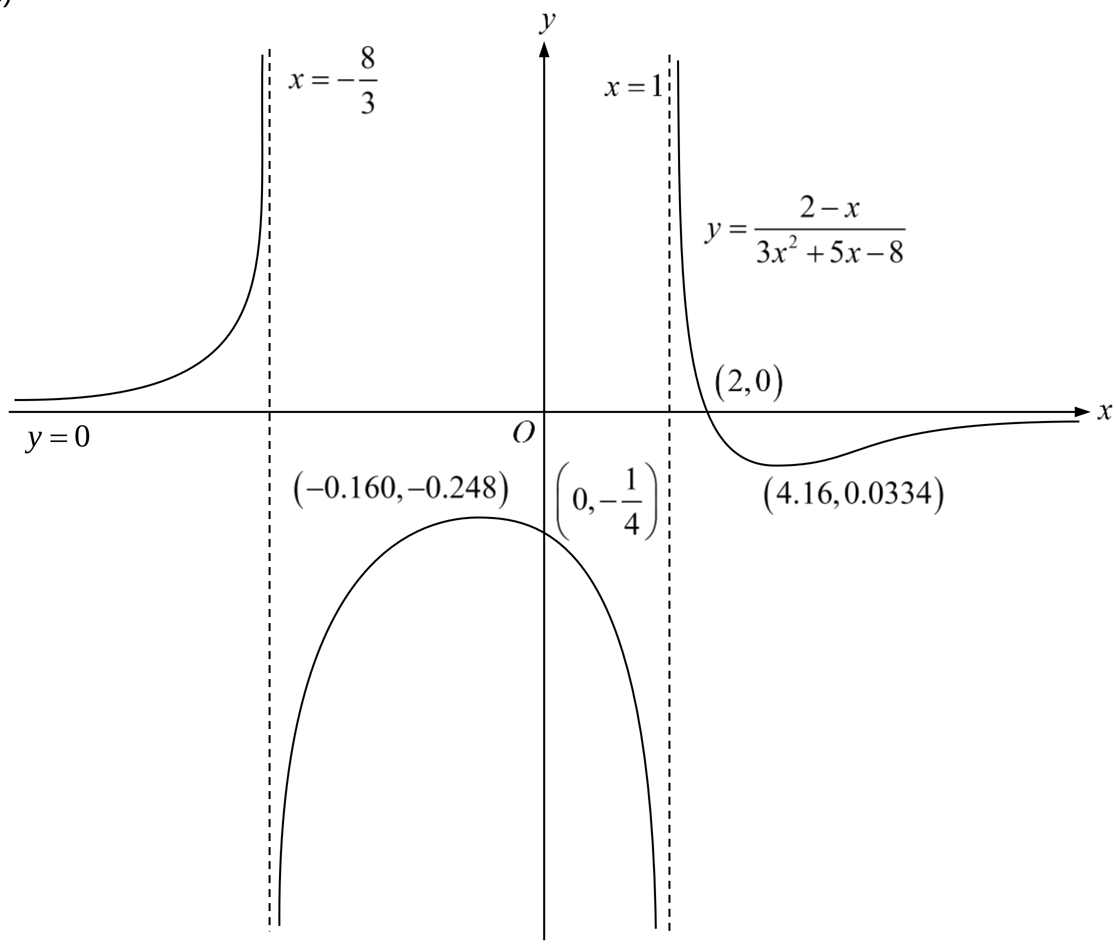
(iii) $\left[-\frac{2}{3}x(1-x)^{\frac{3}{2}}-\frac{4}{15}(1-x)^{\frac{5}{2}}+A\right]-\left[-\frac{2}{3}(1-x)^{\frac{3}{2}}+\frac{2}{5}(1-x)^{\frac{5}{2}}+B\right]$

$$\begin{aligned}
&=\frac{2}{3}(1-x)^{\frac{3}{2}}-\frac{2}{3}x(1-x)^{\frac{3}{2}}-\frac{2}{3}(1-x)^{\frac{5}{2}}+A-B\\
&=\frac{2}{3}(1-x)^{\frac{3}{2}}(1-x)-\frac{2}{3}(1-x)^{\frac{5}{2}}+A-B\\
&=\frac{2}{3}(1-x)^{\frac{5}{2}}-\frac{2}{3}(1-x)^{\frac{5}{2}}+A-B=A-B \text{ (shown)}
\end{aligned}$$

Question 2

[Ans: (i) sketch (ii) $x<-\frac{8}{3}$ or $1<x<2$ (iii) $-\frac{8}{3}<x<1$ or $x>2$]

(i)



(ii) $\frac{2-x}{3x^2+5x-8}>0$

$x<-\frac{8}{3}$ or $1<x<2$

(iii) $\frac{x-2}{3x^2+5x-8}>0$

$\frac{2-x}{3x^2+5x-8}<0$

$-\frac{8}{3}<x<1$ or $x>2$

Question 3

[Ans: $1500\sqrt{\frac{6}{\pi}}$; $1:2$]

Surface area, $A=900$

$2\pi r^2+2\pi rh=900$

$\pi rh=450-\pi r^2$

Volume, $V=\pi r^2 h=r\left(\pi rh\right)$

$V=r\left(450-\pi r^2\right)=450r-\pi r^3$

$\frac{dV}{dr}=450-3\pi r^2$

Let $\frac{dV}{dr}=0$,

$$450-3\pi r^2=0 \Rightarrow r=\sqrt{\frac{150}{\pi}}=5\sqrt{\frac{6}{\pi}}$$

$\frac{d^2V}{dr^2}=-6\pi r$

When $r=5\sqrt{\frac{6}{\pi}}$,

$$\frac{d^2V}{dr^2}=-6\pi\left(5\sqrt{\frac{6}{\pi}}\right)=-30\pi\sqrt{\frac{6}{\pi}}<0$$

$\therefore V$ is a maximum

Max. V

$=450\left(5\sqrt{\frac{6}{\pi}}\right)-\pi\left(5\sqrt{\frac{6}{\pi}}\right)^3$

$=2250\sqrt{\frac{6}{\pi}}-750\sqrt{\frac{6}{\pi}}=1500\sqrt{\frac{6}{\pi}}$

$\pi rh=450-\pi r^2 \Rightarrow h=\frac{450-\pi r^2}{\pi r}$

$\therefore \frac{r}{h}=\frac{r}{\frac{450-\pi r^2}{\pi r}}=\frac{\pi r^2}{450-\pi r^2}$

When V is a maximum, $r=5\sqrt{\frac{6}{\pi}}$,

$$\frac{r}{h}=\frac{\pi\left(5\sqrt{\frac{6}{\pi}}\right)^2}{450-\pi\left(5\sqrt{\frac{6}{\pi}}\right)^2}=\frac{1}{2}\Rightarrow r:h=1:2$$

Question 4

[Ans: (i) $f'(x)=2\sec 2x\tan 2x$; $f''(x)=4\sec 2x\tan^2 2x+4\sec^3 2x$; $f(x)=1+2x^2+\cdots$
(ii) 0.02001 (iii) 0.02001 (iv) comment (v) explain]

(i) $f(x)=\sec 2x$

$$f'(x)=2\sec 2x\tan 2x=2\tan 2x\left[f(x)\right]$$

$$f''(x)=2\tan 2x\left[f'(x)\right]+4\sec^2 2x\left[f(x)\right]$$

$$=2\tan 2x\left(2\sec 2x\tan 2x\right)+4\sec^2 2x\left(\sec 2x\right)$$

$$=4\sec 2x\tan^2 2x+4\sec^3 2x$$

When $x=0$,

$f(0)=1$

$f'(0)=0$

$f''(0)=0+4(1)=4$

$$f(x)=1+(0)x+\frac{4}{2!}x^2+\cdots=1+2x^2+\cdots$$

(ii) $\int_0^{0.02}\sec 2x\,dx$

$\approx\int_0^{0.02}1+2x^2dx=0.02001$ (to 5 decimal places)

(iii) $\int_0^{0.02}\sec 2x\,dx\approx 0.02001$ (to 5 decimal places)

(iv) As the Maclaurin's series is in increasing power of x, the accuracy of its estimation favours values of x closer to zero. Since values of x used in both integrals are close to zero, we can see that the approximations in (ii) is the same as the value in (iii) to 5 decimal places.

(v) As $g(0)$ is undefined, $\therefore$ Maclaurin series for $g(x)$ cannot be found.

Queston 5

[Ans: (i) $\overrightarrow{OX}=\underset{\sim}{b}+\frac{5}{4}\underset{\sim}{a}$ (ii) $\overrightarrow{OY}=\frac{8}{3}\left(\underset{\sim}{b}+\frac{5}{4}\underset{\sim}{a}\right)$; 3:8]

(i) $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\left(2\underset{\sim}{a}+4\underset{\sim}{b}\right)-\underset{\sim}{a}=\underset{\sim}{a}+4\underset{\sim}{b}$

$l_{AC}:\underset{\sim}{r}=\underset{\sim}{a}+\lambda\left(\underset{\sim}{a}+4\underset{\sim}{b}\right)$

$\overrightarrow{BD}=\overrightarrow{OD}-\overrightarrow{OB}=\left(\underset{\sim}{b}+5\underset{\sim}{a}\right)-\underset{\sim}{b}=5\underset{\sim}{a}$

$l_{BD}:\underset{\sim}{r}=\underset{\sim}{b}+\mu\underset{\sim}{a}$

At X,

$\underset{\sim}{a}+\lambda\left(\underset{\sim}{a}+4\underset{\sim}{b}\right)=\underset{\sim}{b}+\mu\underset{\sim}{a}$

$(\lambda+1)\underset{\sim}{a}+4\lambda\underset{\sim}{b}=\underset{\sim}{b}+\mu\underset{\sim}{a}$

$\therefore 4\lambda=1\Rightarrow\lambda=\frac{1}{4}$

$\therefore \mu=\lambda+1=\frac{1}{4}+1=\frac{5}{4}$

$\therefore \overrightarrow{OX}=\underset{\sim}{b}+\frac{5}{4}\underset{\sim}{a}$

(ii) $\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}=\left(\underset{\sim}{b}+5\underset{\sim}{a}\right)-\left(2\underset{\sim}{a}+4\underset{\sim}{b}\right)=3\underset{\sim}{a}-3\underset{\sim}{b}=3\left(\underset{\sim}{a}-\underset{\sim}{b}\right)$

$l_{CD}:\underset{\sim}{r}=\left(2\underset{\sim}{a}+4\underset{\sim}{b}\right)+\gamma\left(\underset{\sim}{a}-\underset{\sim}{b}\right)$

$l_{OX}:\underset{\sim}{r}=\sigma\left(\underset{\sim}{b}+\frac{5}{4}\underset{\sim}{a}\right)$

At Y,

$\left(2\underset{\sim}{a}+4\underset{\sim}{b}\right)+\gamma\left(\underset{\sim}{a}-\underset{\sim}{b}\right)=\sigma\left(\underset{\sim}{b}+\frac{5}{4}\underset{\sim}{a}\right)$

$(2+\gamma)\underset{\sim}{a}+(4-\gamma)\underset{\sim}{b}=\frac{5}{4}\sigma\underset{\sim}{a}+\sigma\underset{\sim}{b}$

$\therefore 2+\gamma=\frac{5}{4}\sigma\Rightarrow 4\gamma-5\sigma=-8$

$\therefore 4-\gamma=\sigma\Rightarrow\gamma+\sigma=4$

From GC, $\gamma=\frac{4}{3}$, $\sigma=\frac{8}{3}$

$\therefore \overrightarrow{OY}=\frac{8}{3}\left(\underset{\sim}{b}+\frac{5}{4}\underset{\sim}{a}\right)$

$\overrightarrow{OY}=\frac{8}{3}\overrightarrow{OX}\Rightarrow OY=\frac{8}{3}OX\Rightarrow\frac{OX}{OY}=\frac{3}{8}$

$\therefore OX:OY=3:8$

Question 6

[Ans: (i) explain (ii) explain (iii) 7.24×10^{18}]

(i) These 22 clubs form a population as they are all the clubs in Division One.

(ii) Dilip can label each club from 1 to 100 and investigate all the even-labelled clubs. This is easy to implement and it will ensure that it is an evenly spread investigation amongst the 100 clubs.

(iii) Number of different samples
$= {}^{22}C_5\,{}^{24}C_5\,{}^{26}C_5\,{}^{28}C_5 = 7.24\times10^{18}$

Question 7

[Ans: (i) state assumptions (ii) 0.102 (iii) 0.991 (iv) $45p^2(1-p)^8$

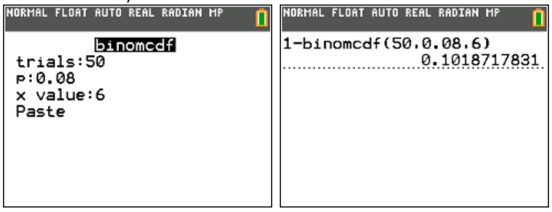
(v) $1.6928p(1-p)+0.9936(1-p)^2=0.97$; 0.0689]

(i) Assumption (1):
Assume that any mug being faulty is independent of any other mugs.
Assumption (2):
Assume that the probability of any randomly chosen mug being faulty is always 8%.

(ii) $F \sim B(50, 0.08)$

$P(F \geq 7) = 1 - P(F \leq 6) = 0.10187 \approx 0.102$

From GC,

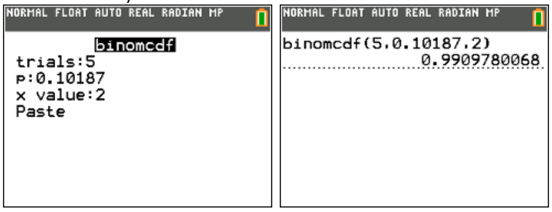


(iii) Let X be the number of days that at least 7 faulty mugs are found out of 5 working days.

$X \sim B(5, 0.10187)$

$P(X \leq 2) = 0.99098 \approx 0.991$

From GC,



(iv) Let Y be the number of faulty saucers out of 10 saucers made.

$Y \sim B(10, p)$

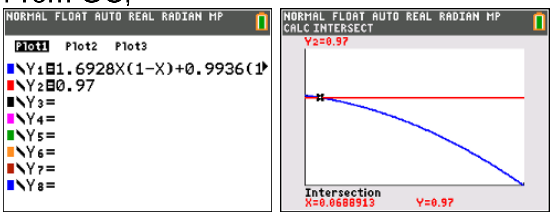
$$P(Y=2) = \binom{10}{2} p^2 (1-p)^{10-2} = 45p^2(1-p)^8$$

(v) Probability that a set contains at most 1 faulty item $= 0.97$

$$(1-0.08)^2\left[\binom{2}{1}p(1-p)\right] + \left[\binom{2}{1}(0.08)(1-0.08)\right](1-p)^2 + (1-0.08)^2(1-p)^2 = 0.97$$

$$1.6928p(1-p) + 0.9936(1-p)^2 = 0.97$$

From GC,



$p = 0.068891 \approx 0.0689$

Question 8

[Ans: (i)(a) $\frac{23}{56}$ (b) $\frac{13}{28}$ (ii)(a) $\frac{1}{55}$ (b) $\frac{21}{110}$ (iii) possible combinations]

(i) (a) Required probability

$$=\frac{(1+1+3+4)+(1+1+7+5)}{56}=\frac{23}{56}$$

(b) Required probability

$$=\frac{(3+7+1)+(4+5+6)}{56}=\frac{13}{28}$$

(ii) (a) Required probability

$$=\left(\frac{8}{56}\right)\left(\frac{7}{55}\right)=\frac{1}{55}$$

(b) Required probability

$$=\left(\frac{7}{56}\right)\left(\frac{32}{55}\right)(2!)+\left(\frac{10}{56}\right)\left(\frac{7}{55}\right)(2!)=\frac{21}{110}$$

(iii) Let $x=1,3,4,5,6$ or 7 and $y=1,3,4,5,6$ or 7

$$\left(\frac{x}{56}\right)\left(\frac{y}{55}\right)(2!)=\frac{1}{77}$$

$xy=20$

$x=4$ or 5 and $y=4$ or 5

$\therefore$ all possible combinations for Gerri's two favourite colours/characters are:
White Horse and White Rider,
White Horse and Yellow Bird,
White Rider and Orange Bird, and
Orange Bird and Yellow Bird

Question 9

[Ans: (i) explain; state (ii) 756; p-value $=0.089686$, insufficient evidence to reject (iii) describe]

(i) Since the production manager wishes to test if the mean resistance is not 750 ohms, a 2-tail test should be carried out.

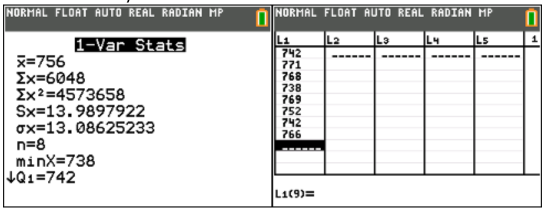
Let μ be the mean resistance, H_0 be the null hypothesis, and H_1 be the alternative hypothesis.

$H_0 : \mu = 750$

$H_1 : \mu \neq 750$

(ii) Let X be the resistance of a randomly chosen resistor.

From GC,



$n=8$

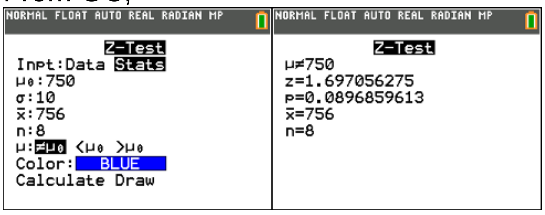
$\bar{x}=756$

$\sigma^2=100$

Test Statistics,

$$\bar{X} \sim N\left(750, \frac{100}{8}\right) \Rightarrow Z = \frac{\bar{X}-750}{\sqrt{100/8}} \sim N(0,1)$$

From GC,



p-value $=0.089686$

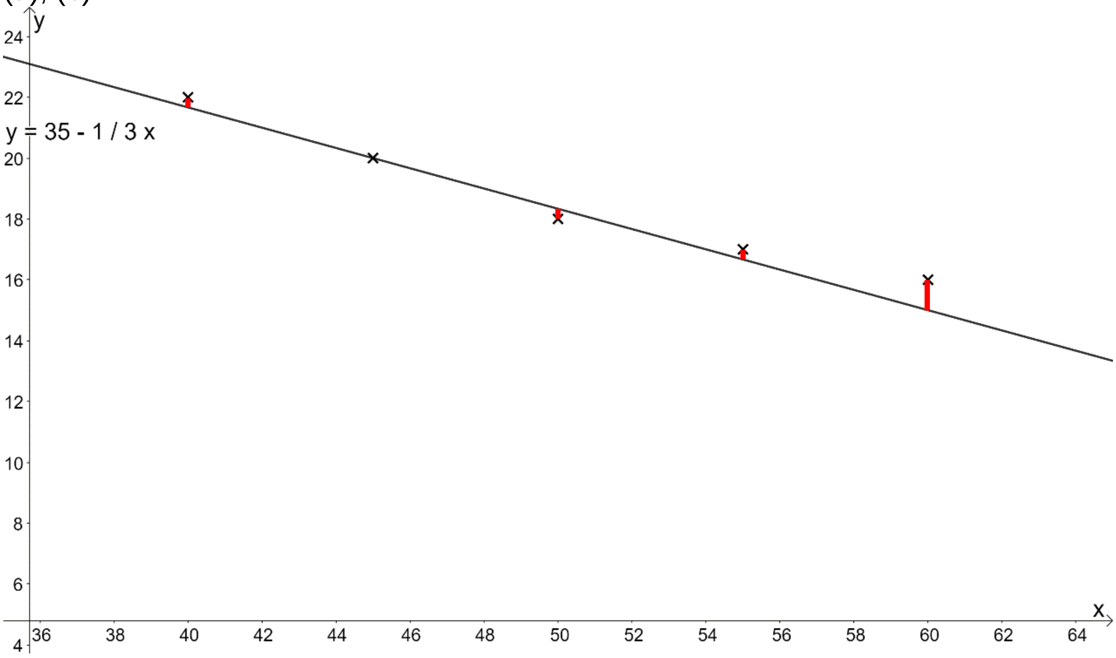
Since p-value >0.05, there is insufficient evidence that the mean resistance of these resistors is not 750 ohms at 5% level of significance.

(iii) Since nothing is known about the distribution of the resistances of these resistors, a large sample size will be required so as to apply Central Limit Theorem to approximate the sample mean distribution for the test statistics to a normal distribution.

Question 10

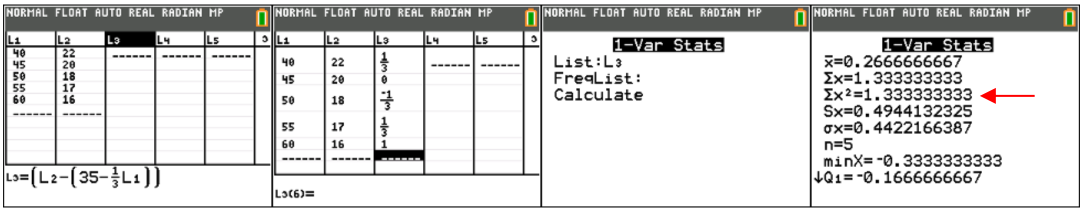
[Ans: (i)(a) draw (b) mark (c) 1.33 (d) explain (ii) Bhani (iii) $(50, 18.6)$
(iv) $y = 33.6 - 0.3x$, $r = -0.985$ (v) 24.6 km/l; explain (vi) deduce]

(i) (a), (b)



(b) *On the graph, [|] represents the residual.*

(c) From GC, letting L1 $= x$, L2 $= y$ and L3 $= 35 - \frac{1}{3}x$

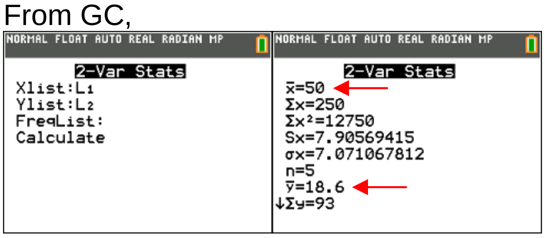


Sum of square of the residuals $= 1.33$

(d) Residuals calculated can be positive or negative, and if they are sum together it will not be possible to gauge the absolute residuals as these values may cancel out each other.

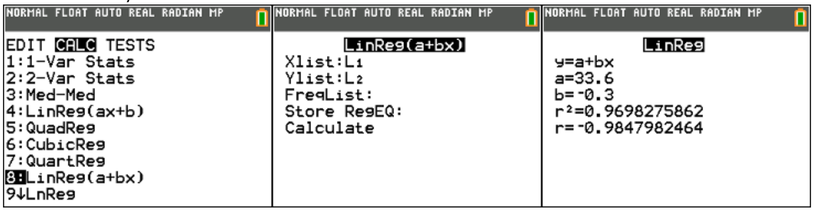
(ii) Since the calculated sum of squares of the residual for Bhani's line is smaller, Bhani's model gives a better fit.

(iii) From GC,



$(50, 18.6)$

(iv) From GC,



Least square regression line: $y = 33.6 - 0.3x$

$r = -0.985$

(v) When $x = 30$,

$y = 33.6 - 0.3(30) = 24.6$

As $x = 30$ is outside the data range, $40 \leq x \leq 60$, the estimated fuel consumption is unreliable.

(vi) Cerie's data points are perfectly collinear.

Question 11

[Ans: (i) 0.970 (ii) 0.802 (iii) 381 (iv) 0.846]

Let X be the mass of a randomly chosen white ball, and Y be the mass of a randomly chosen black ball.

$X \sim N\left(110, 4^2\right), \qquad Y \sim N\left(55, 2^2\right)$

(i) $E\left(X_1 + \cdots + X_4\right) = 4E\left(X\right) = 4\left(110\right) = 440$

$Var\left(X_1 + \cdots + X_4\right) = 4Var\left(X\right) = 4\left(4^2\right) = 64$

$X_1 + \cdots + X_4 \sim N\left(440, 64\right)$

$P\left(X_1 + \cdots + X_4 > 425\right) = 0.96960 \approx 0.970$

(ii) $E\left(X + Y\right) = E\left(X\right) + E\left(Y\right) = 110 + 55 = 165$

$Var\left(X + Y\right) = Var\left(X\right) + Var\left(Y\right) = 4^2 + 2^2 = 20$

$X + Y \sim N\left(165, 20\right)$

$P\left(161 < X + Y < 175\right) = 0.80178 \approx 0.802$

(iii) $E\left(X_1 + X_2 + Y_1 + Y_2 + Y_3\right) = 2E\left(X\right) + 3E\left(Y\right) = 2\left(110\right) + 3\left(55\right) = 385$

$Var\left(X_1 + X_2 + Y_1 + Y_2 + Y_3\right) = 2Var\left(X\right) + 3Var\left(Y\right) = 2\left(4^2\right) + 3\left(2^2\right) = 44$

$X_1 + X_2 + Y_1 + Y_2 + Y \sim N\left(385, 44\right)$

$P\left(X_1 + X_2 + Y_1 + Y_2 + Y < M\right) = 0.271$

$M = 380.96 \approx 381$

(iv) Let W be the mass of a randomly chosen connecting rod.

$W \sim N\left(20, 0.9^2\right)$

$E\left(0.9\left(Y_1 + \cdots + Y_4\right) + 0.7X + \left(W_1 + \cdots + W_4\right)\right)$

$= 0.9\left[4E\left(Y\right)\right] + 0.7E\left(X\right) + 4E\left(W\right)$

$= 3.6\left(55\right) + 0.7\left(110\right) + 4\left(20\right) = 355$

$Var\left(0.9\left(Y_1 + \cdots + Y_4\right) + 0.7X + \left(W_1 + \cdots + W_4\right)\right)$

$= 0.9^2\left[4Var\left(Y\right)\right] + 0.7^2 Var\left(X\right) + 4Var\left(W\right)$

$= 3.24\left(2^2\right) + 0.49\left(4^2\right) + 4\left(0.9^2\right) = 24.04$

$0.9\left(Y_1 + \cdots + Y_4\right) + 0.7X + \left(W_1 + \cdots + W_4\right) \sim N\left(355, 24.04\right)$

$P\left(0.9\left(Y_1 + \cdots + Y_4\right) + 0.7X + \left(W_1 + \cdots + W_4\right) > 350\right) = 0.84608 \approx 0.846$