A-LEVEL H2 MATHS 2019 – PAPER 1

Question 1

[Ans: $b=-a$, $c=-7a$ and $d=15a$]

Given $f(z)=az^3+bz^2+cz+d$

Since roots of $f(z)=0$ are $2+i$, $2-i$ and -3 ,

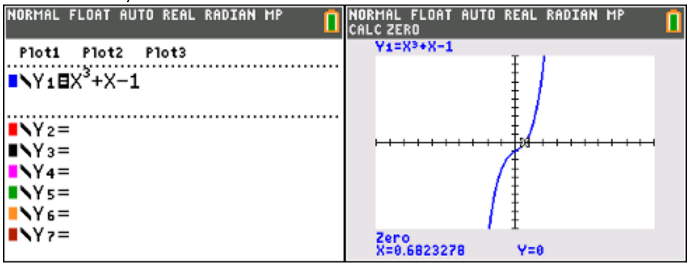
$$\begin{aligned} f(z) &= a\left[z-(2+i)\right]\left[z-(2-i)\right](z+3) \\ &= a\left[z^2-2(2)z+\left(2^2+1^2\right)\right](z+3) \\ &= a\left(z^2-4z+5\right)(z+3) \\ &= a\left(z^3+3z^2-4z^2-12z+5z+15\right) \\ &= a\left(z^3-z^2-7z+15\right)=az^3-az^2-7az+15a \end{aligned}$$

$\therefore b=-a$, $c=-7a$ and $d=15a$

Question 2

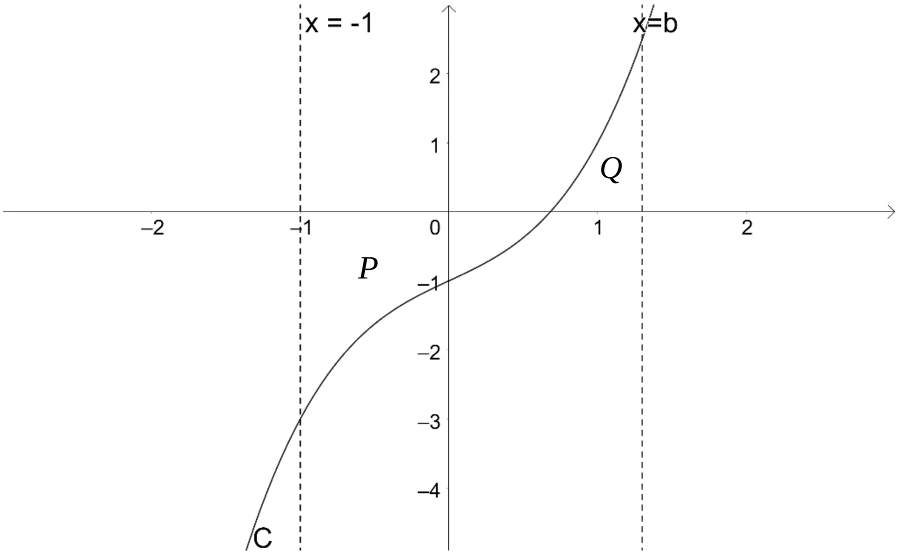
[Ans: (i) 0.682 (ii) 1.892]

(i) From GC,



$a = 0.682$

(ii)



Area of P

$$= -\int_{-1}^{0} x^3 + x - 1 dx = \frac{7}{4}$$

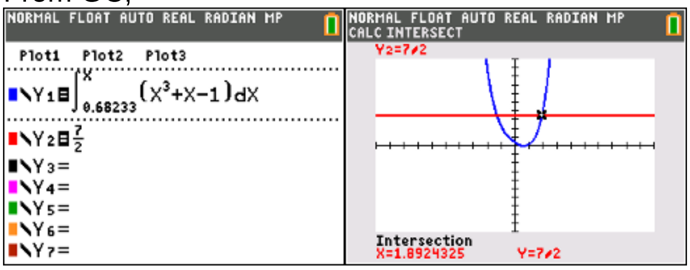
Area of Q

$$= \int_{0.68233}^{b} x^3 + x - 1 dx$$

Area of $Q = 2(\text{Area of } P)$

$$\int_{0.68233}^{b} x^3 + x - 1 dx = 2\left(\frac{7}{4}\right) = \frac{7}{2}$$

From GC,



$b = 1.892$

Question 3

[Ans: (i) $p=2$, $q=-1$, $r=-5$ (ii) describe]

(i) $f(x)$

$=2x^3-6x^2+6x-12$

$=2\left[\left(x^3-3x^2+3x-1\right)-5\right]$

$=2\left[(x-1)^3-5\right]$

$\therefore p=2$, $q=-1$, $r=-5$

(ii) Transformation (1): Translate by 1 unit in the positive x-direction
Transformation (2): Translate by 5 units in the negative y-direction
Transformation (3): Scale by a factor of 2 parallel to the y-axis

Question 4

[Ans: (i) sketch (ii) $x\in[2,4]$]

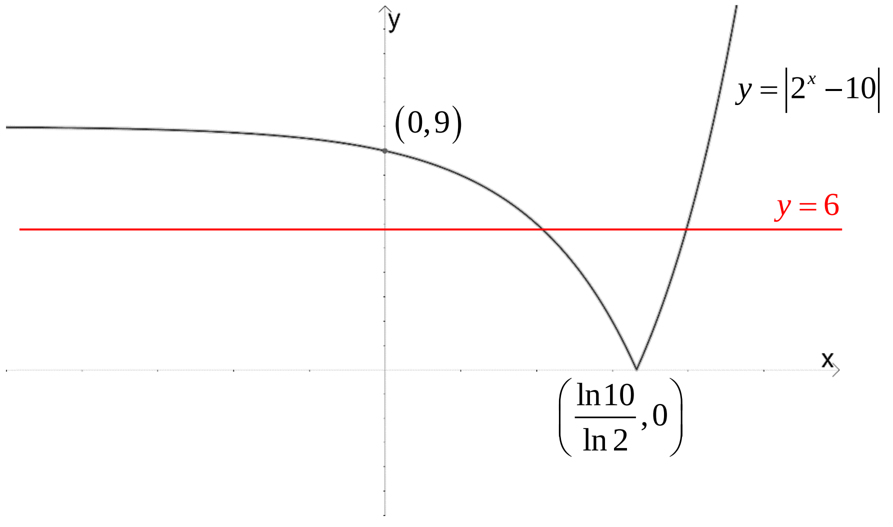
(i) When $x=0$,

$y=\left|2^0-10\right|=9$

When $y=0$,

$\left|2^x-10\right|=0$

$2^x=10 \Rightarrow x\ln 2=\ln 10 \Rightarrow x=\dfrac{\ln 10}{\ln 2}$



(ii) Let $\left|2^x-10\right|=6$

$2^x-10=-6$ or $2^x-10=6$

$2^x=4$; $2^x=16$

$x=2$; $x=4$

Observing from the graph in (i), for $\left|2^x-10\right|\le 6$, $x\in[2,4]$.

Queston 5

[Ans: (i) $f^{-1}(x)=\frac{1}{2}\ln(x+4)$; $D_{f^{-1}}=(-4,\infty)$ (ii) $\ln 3-2$]

(i) Let $y=f(x)$

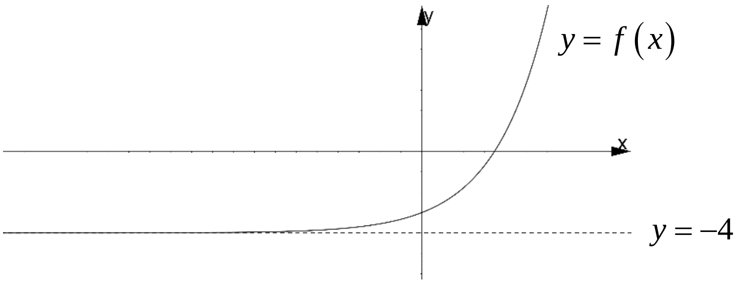
$y=e^{2x}-4$, $x\in\mathbb{R}$

$e^{2x}=y+4$

$2x=\ln(y+4)$

$x=\frac{1}{2}\ln(y+4)$

$\therefore f^{-1}(x)=\frac{1}{2}\ln(x+4)$



$D_{f^{-1}}=R_f=(-4,\infty)$

(ii) $fg(x)=5$

$g(x)=f^{-1}(5)$

$x+2=\frac{1}{2}\ln(5+4)=\frac{1}{2}\ln 9$

$x=\ln 3-2$

Question 6

[Ans: (i) $\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$ (ii) $\frac{1}{42}$]

(i) Let $\frac{1}{4r^2-1}=\frac{1}{(2r-1)(2r+1)}=\frac{A}{2r-1}+\frac{B}{2r+1}$

$A=\frac{1}{2\left(\frac{1}{2}+1\right)}=\frac{1}{2}$

$B=\frac{1}{2\left(-\frac{1}{2}\right)-1}=-\frac{1}{2}$

$\therefore \frac{1}{4r^2-1}=\frac{1}{2(2r-1)}-\frac{1}{2(2r+1)}=\frac{1}{2}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$

$$\sum_{r=1}^{n}\frac{1}{4r^2-1}$$
$$=\sum_{r=1}^{n}\frac{1}{2}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)=\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$$
$$=\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right.$$
$$+\frac{1}{3}-\frac{1}{5}$$
$$+\frac{1}{5}-\frac{1}{7}$$
$$\vdots$$
$$+\frac{1}{2n-3}-\frac{1}{2n-1}$$
$$\left.+\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$
$$=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$$

(ii) $\sum_{r=11}^{n}\frac{1}{4r^2-1}$

$$=\sum_{r=1}^{\infty}\frac{1}{4r^2-1}-\sum_{r=1}^{10}\frac{1}{4r^2-1}$$
$$=\frac{1}{2}(1-0)-\frac{1}{2}\left[1-\frac{1}{2(10)+1}\right]=\frac{1}{42}$$

Question 7

[Ans: (i) $y=\frac{1}{e}$; $y=e\left(2x+1\right)$ (ii) $\tan^{-1}\left(2e\right)$]

(i) $\frac{dy}{dx}=x\left(-e^{x}\right)+\left(1\right)e^{-x}=e^{-x}\left(1-x\right)$

When $x=1$,

$y=e^{-1}=\frac{1}{e}$

$\frac{dy}{dx}=e^{-1}\left(1-1\right)=0$

Equation of tangent: $y=\frac{1}{e}$

When $x=-1$,

$y=\left(-1\right)e^{-\left(-1\right)}=-e$

$\frac{dy}{dx}=e^{-\left(-1\right)}\left[1-\left(-1\right)\right]=2e$

Equation of tangent:

$y-\left(-e\right)=2e\left[x-\left(-1\right)\right]$

$y=2ex+e=e\left(2x+1\right)$

(ii) Acute angle between the tangents

$=\tan^{-1}\left(2e\right)$

Question 8

[Ans: (a) $k=13$ (b) any real number other than 0 can be accepted as possible value for f, $r=-1$; Sum $=0$ or f (c) 63]

(a) $a(2)^{k-1}=\frac{64}{2}\left[2a+(64-1)(2a)\right]$

$2^{k-1}=4096=2^{12}\Rightarrow k=13$

(b) $\frac{f\left(1-r^4\right)}{1-r}=0$

$f\left(1-r^4\right)=0$

Since $f\neq 0$,

$1-r^4=0\Rightarrow r=-1$ or $r=1$ (NA)

$\therefore f$ can be any real number except 0, e.g. $f=5$, and $r=-1$

Sum of first n terms

$=\frac{f\left[1-(-1)^n\right]}{1-(-1)}=\frac{1}{2}f\left[1-(-1)^n\right]$

$=\begin{cases}0 & n=2k, k\in\mathbb{Z}^+\\ f & n=2k+1, k\in\mathbb{Z}_0^+\end{cases}$

(c) Let the first term and common difference of the arithmetic series be A and D.

$\frac{4}{2}\left[2A+(4-1)D\right]=14\Rightarrow 2A+3D=7$

$A(A+D)(A+2D)(A+3D)=0$

$A\neq 0$ ($\because A<0$)

If $A+D=0\Rightarrow D=-A$,

$2A+3(-A)=7\Rightarrow A=-7$

If $A+2D=0\Rightarrow D=-\frac{1}{2}A$,

$2A+3\left(-\frac{1}{2}A\right)=7\Rightarrow A=14$ (NA $\because A<0$)

If $A+3D=0\Rightarrow D=-\frac{1}{3}A$,

$2A+3\left(-\frac{1}{3}A\right)=7\Rightarrow A=7$ (NA $\because A<0$)

$\therefore A=-7$ and $D=7$

11th term of the series $=A+(11-1)D=-7+(10)(7)=63$

Question 9

[Ans: (i)(a) show (b) show; $k=i$ (ii) 1]

(i) $w=\cos\theta+i\sin\theta=e^{i\theta}$

(a) $w+\dfrac{1}{w}$

$=e^{i\theta}+\dfrac{1}{e^{i\theta}}=e^{i\theta}+e^{-i\theta}$

$=\left(\cos\theta+i\sin\theta\right)+\left(\cos\theta-i\sin\theta\right)$

$=2\cos\theta\in\mathbb{R}$ (shown)

(b) $\dfrac{w-1}{w+1}$

$$=\frac{e^{i\theta}-1}{e^{i\theta}+1}=\frac{e^{i\frac{1}{2}\theta}\left(e^{i\frac{1}{2}\theta}-e^{-i\frac{1}{2}\theta}\right)}{e^{i\frac{1}{2}\theta}\left(e^{i\frac{1}{2}\theta}+e^{-i\frac{1}{2}\theta}\right)}$$

$$=\frac{\left(\cos\frac{1}{2}\theta+i\sin\frac{1}{2}\theta\right)-\left(\cos\frac{1}{2}\theta-i\sin\frac{1}{2}\theta\right)}{\left(\cos\frac{1}{2}\theta+i\sin\frac{1}{2}\theta\right)+\left(\cos\frac{1}{2}\theta-i\sin\frac{1}{2}\theta\right)}$$

$$=\frac{2i\sin\frac{1}{2}\theta}{2\cos\frac{1}{2}\theta}=i\tan\frac{1}{2}\theta$$ (shown), $k=i$

(ii) Let $z=x+iy$ and $|z|=1\Rightarrow x^2+y^2=1$

$$\frac{z-3i}{1+3iz}=\frac{x+iy-3i}{1+3i(x+iy)}$$

$$=\frac{x+i(y-3)}{(1-3y)+3ix}\left[\frac{(1-3y)-3ix}{(1-3y)-3ix}\right]$$

$$=\frac{x(1-3y)-3ix^2+i(y-3)(1-3y)+3x(y-3)}{(1-3y)^2+(3x)^2}$$

$$=\frac{\left[(x-3xy)+(3xy-9x)\right]+i\left[-3x^2+\left(y-3y^2-3+9y\right)\right]}{\left(1-6y+9y^2\right)+9x^2}$$

$$=\frac{-8x+i\left[-3\left(x^2+y^2\right)-3+10y\right]}{1-6y+9\left(x^2+y^2\right)}$$

$$=\frac{-8x+i(10y-6)}{10-6y}=\frac{-8x}{10-6y}+i\frac{10y-6}{10-6y}$$

$$\left|\frac{z-3i}{1+3iz}\right| = \sqrt{\left(\frac{-8x}{10-6y}\right)^2 + \left(\frac{10y-6}{10-6y}\right)^2}$$

$$= \sqrt{\frac{64x^2+100y-120y+36}{(10-6y)^2}}$$

$$= \sqrt{\frac{64\left(1-y^2\right)+100y-120y+36}{(10-6y)^2}}$$

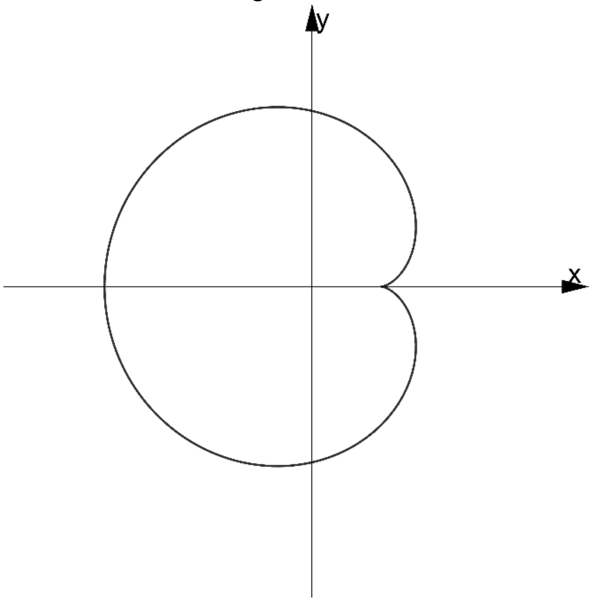
$$= \sqrt{\frac{100-120y+36y^2}{(10-6y)^2}}$$

$$= \sqrt{\frac{(10-6y)^2}{(10-6y)^2}} = 1$$

Question 10

[Ans: (i) sketch; $y=0$ (ii) $\theta=0,\pi,2\pi$ (iii) show; $\theta_1=0$, $\theta_2=\pi$ (iv) $6\pi a^2$]

(i) From GC, assuming $a>0$,



Line of symmetry: $y=0$

(ii) Let $y=0$,

$a\left(2\sin\theta-\sin2\theta\right)=0$

$2\sin\theta-2\sin\theta\cos\theta=0$

$\sin\theta\left(1-\cos\theta\right)=0$

$\sin\theta=0$ or $\cos\theta=1$

$\theta=0,\pi,2\pi$ $\theta=0,2\pi$

(iii) When $\theta=0$, $x=a\left(2-1\right)=a$

When $\theta=\pi$, $x=a\left(-2-1\right)=-3a$

When $\theta=2\pi$, $x=a\left(2-1\right)=a$

Let $\theta=\dfrac{\pi}{2}$,

$x=a\left(2\cos\dfrac{\pi}{2}-\cos\pi\right)=a$, $y=a\left(2\sin\dfrac{\pi}{2}-\sin\pi\right)=2a$

Assuming $a>0$, $\left(a,2a\right)$ will above the x-axis.

$\therefore$ Area required

$=\int_{-3a}^{a}y\,dx$

$=\int_{\pi}^{0}a\left(2\sin\theta-\sin2\theta\right)a\left(-2\sin\theta+2\sin2\theta\right)d\theta$

$=\int_{\pi}^{0}a^2\left(-4\sin^2\theta+4\sin\theta\sin2\theta+2\sin\theta\sin2\theta-2\sin^2 2\theta\right)d\theta$

$=\int_{0}^{\pi}a^2\left(4\sin^2\theta-6\sin\theta\sin2\theta+2\sin^2 2\theta\right)d\theta$ (shown), where $\theta_1=0$, $\theta_2=\pi$

(iv) Total area enclosed by C

$$= 2\int_0^{\pi} a^2 \left(4\sin^2\theta - 6\sin\theta\sin 2\theta + 2\sin^2 2\theta\right) d\theta$$

$$= 4a^2 \int_0^{\pi} 2\sin^2\theta - 3\sin\theta\sin 2\theta + \sin^2 2\theta d\theta$$

$$= 4a^2 \int_0^{\pi} 1 - \cos 2\theta - 3\sin\theta\left(2\sin\theta\cos\theta\right) + \frac{1-\cos 4\theta}{2} d\theta$$

$$= 4a^2 \int_0^{\pi} 1 - \cos 2\theta - 6\left(\cos\theta\right)\left(\sin\theta\right)^2 + \frac{1}{2} - \frac{1}{2}\cos 4\theta d\theta$$

$$= 4a^2 \left[\theta - \frac{1}{2}\sin 2\theta - 2\sin^3\theta + \frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\right]_0^{\pi}$$

$$= 4a^2 \left[\left(\pi - 0 - 0 + \frac{1}{2}\pi - 0\right) - \left(0 - 0 - 0 + 0 - 0\right)\right] = 4a^2\left(\frac{3}{2}\pi\right) = 6\pi a^2$$

Question 11

[Ans: (i)(a) $\theta = 16 + 64(4)^{-\frac{t}{30}}$ (b) $24°C$ (ii) 540 min]

(i) (a) Let $\frac{d\theta}{dt} = -k(\theta - 16)$, $k > 0$

$$\frac{1}{\theta - 16}\frac{d\theta}{dt} = -k$$

$$\int \frac{1}{\theta - 16}\, d\theta = -k\int dt$$

$$\ln|\theta - 16| = -kt + A$$

$$|\theta - 16| = e^{-kt+A} = e^{A}e^{-kt}$$

$$\theta - 16 = Be^{-kt},\ B = \pm e^{A}$$

$$\theta = 16 + Be^{-kt}$$

When $t = 0$,

$$\theta = 80$$

$$16 + B = 80 \Rightarrow B = 64$$

When $t = 30$,

$$\theta = 32$$

$$16 + 64e^{-30k} = 32$$

$$e^{-30k} = 4^{-1} \Rightarrow e^{-k} = 4^{-\frac{1}{30}}$$

$$\therefore \theta = 16 + 64e^{-kt} = 16 + 64\left(e^{-k}\right)^{t} = 16 + 64(4)^{-\frac{t}{30}}$$

(b) When $t = 45$,

$$\theta = 16 + 64(4)^{-\frac{45}{30}} = 24°C$$

(ii) $\frac{dT}{dt} = \frac{h}{T}$, $h > 0$

$$\frac{1}{h}T\frac{dT}{dt} = 1$$

$$\frac{1}{h}\int T\,dT = \int dt \Rightarrow t = \frac{1}{2h}T^{2} + C$$

When $t = 0$, $T = 0$,

$$0 = 0 + C \Rightarrow C = 0$$

When $t = 60$, $T = 1$,

$$60 = \frac{1}{2h}1^{2} \Rightarrow h = \frac{1}{120}$$

$$\therefore t = \frac{1}{2\left(\frac{1}{120}\right)}T^{2} = 60T^{2}$$

When it is safe to skate, $T = 3$, $\therefore t = 60\left(3^{2}\right) = 540$

Question 12

[Ans: (i) $Q\left(\frac{8}{11},\frac{1}{11},\frac{2}{11}\right)$, $R\left(-\frac{37}{11},-\frac{39}{11},-\frac{23}{11}\right)$ (ii) $\cos\theta=\frac{11}{7\sqrt{3}}$, $\cos\beta=\frac{22}{\sqrt{510}}$ (iii) $\frac{10}{\sqrt{3}}$ (iv) $\frac{\sqrt{170}}{7}$ (v) $k<1$]

(i) $l_{QR}: \underset{\sim}{r}=\begin{pmatrix}2\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}-2\\-3\\-6\end{pmatrix}$; Plane containing top of prism: $\underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=1$

At Q,

$$\begin{pmatrix}2-2\lambda\\2-3\lambda\\4-6\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=1$$

$$2-2\lambda+2-3\lambda+4-6\lambda=1\Rightarrow\lambda=\frac{7}{11}$$

$$\therefore\overrightarrow{OQ}=\begin{pmatrix}2\\2\\4\end{pmatrix}+\frac{7}{11}\begin{pmatrix}-2\\-3\\-6\end{pmatrix}=\begin{pmatrix}8/11\\1/11\\2/11\end{pmatrix}$$

$$Q\left(\frac{8}{11},\frac{1}{11},\frac{2}{11}\right)$$

$l_{RS}: \underset{\sim}{r}=\begin{pmatrix}-5\\-6\\-7\end{pmatrix}+\mu\begin{pmatrix}-2\\-3\\-6\end{pmatrix}$; Plane containing base of prism: $\underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=-9$

At R,

$$\begin{pmatrix}-5-2\mu\\-6-3\mu\\-7-6\mu\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=-9$$

$$-5-2\mu-6-3\mu-7-6\mu=-9\Rightarrow\mu=-\frac{9}{11}$$

$$\therefore\overrightarrow{OR}=\begin{pmatrix}-5\\-6\\-7\end{pmatrix}+\left(-\frac{9}{11}\right)\begin{pmatrix}-2\\-3\\-6\end{pmatrix}=\begin{pmatrix}-37/11\\-39/11\\-23/11\end{pmatrix}$$

$$R\left(-\frac{37}{11},-\frac{39}{11},-\frac{23}{11}\right)$$

(ii) $\left|\begin{pmatrix}-2\\-3\\-6\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right|=\left|\begin{pmatrix}-2\\-3\\-6\end{pmatrix}\right|\left|\begin{pmatrix}1\\1\\1\end{pmatrix}\right|\cos\theta$

$$\cos\theta=\frac{|-11|}{\sqrt{49}\sqrt{3}}=\frac{11}{7\sqrt{3}}$$

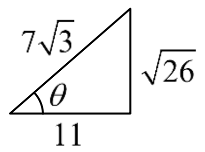
$$\overrightarrow{QR}=\begin{pmatrix}-37/11\\-39/11\\-23/11\end{pmatrix}-\begin{pmatrix}8/11\\1/11\\2/11\end{pmatrix}=\begin{pmatrix}-45/11\\-40/11\\-25/11\end{pmatrix}=-\frac{5}{11}\begin{pmatrix}9\\8\\5\end{pmatrix}$$

$$\left|\begin{pmatrix}9\\8\\5\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right|=\left|\begin{pmatrix}9\\8\\5\end{pmatrix}\right|\left|\begin{pmatrix}1\\1\\1\end{pmatrix}\right|\cos\beta$$

$$\cos\beta=\frac{|22|}{\sqrt{170}\sqrt{3}}=\frac{22}{\sqrt{510}}$$

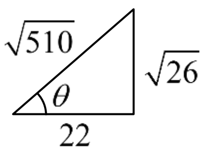
(iii) Thickness $=\left|\dfrac{1}{\left|\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}-\dfrac{-9}{\left|\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}\right|=\dfrac{10}{\sqrt{3}}$

(iv) $\cos\theta=\dfrac{11}{7\sqrt{3}}$



$\sin\theta=\dfrac{\sqrt{26}}{7\sqrt{3}}$

$$\cos\beta=\frac{22}{\sqrt{510}}$$



$\sin\beta=\dfrac{\sqrt{26}}{\sqrt{510}}$

$$\sin\theta=k\sin\beta\Rightarrow k=\frac{\left(\frac{\sqrt{26}}{7\sqrt{3}}\right)}{\left(\frac{\sqrt{26}}{\sqrt{510}}\right)}=\frac{\sqrt{510}}{7\sqrt{3}}=\frac{\sqrt{170}}{7}$$

(v) Since sine is an increasing function when the angle is acute, if $\beta>\theta$, then
$\sin\beta>\sin\theta$

$\therefore\dfrac{\sin\theta}{\sin\beta}<1\Rightarrow k<1$