O-LEVEL A-MATHS 2019 – PAPER 2

Question 1

[Ans: (i) $x\left(1+2\ln x\right)$ (ii) $\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$]

(i) $\frac{d}{dx}\left(x^2\ln x\right)$

$=x^2\left(\frac{1}{x}\right)+2x\ln x=x+2x\ln x=x\left(1+2\ln x\right)$

(ii) $\frac{d}{dx}\left(x^2\ln x\right)=x+2x\ln x$

$x^2\ln x=\int x+2x\ln x\,dx$

$x^2\ln x=\int x\,dx+2\int x\ln x\,dx$

$x^2\ln x=\frac{1}{2}x^2+2\int x\ln x\,dx$

$2\int x\ln x\,dx=x^2\ln x-\frac{1}{2}x^2$

$\int x\ln x\,dx=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$

Question 2

[Ans: (i) prove (ii) $\theta=30^\circ$ or 150°]

(i) LHS

$=\frac{\sin\theta}{1-\cos\theta}-\frac{1}{\sin\theta}$

$=\frac{\sin^2\theta-\left(1-\cos\theta\right)}{\sin\theta\left(1-\cos\theta\right)}$

$=\frac{\left(1-\cos^2\theta\right)-\left(1-\cos\theta\right)}{\sin\theta\left(1-\cos\theta\right)}=\frac{\left(1-\cos\theta\right)\left(1+\cos\theta\right)-\left(1-\cos\theta\right)}{\sin\theta\left(1-\cos\theta\right)}$

$=\frac{\left(1-\cos\theta\right)\left(1+\cos\theta-1\right)}{\sin\theta\left(1-\cos\theta\right)}=\frac{\cos\theta}{\sin\theta}=\cot\theta=$ RHS (proven)

(ii) $\frac{\sin\theta}{1-\cos\theta}-\frac{1}{\sin\theta}=3\tan\theta$

$\cot\theta=3\tan\theta$

$\tan^2\theta=\frac{1}{3}$

$\tan\theta=-\frac{1}{\sqrt{3}}$ or $\tan\theta=\frac{1}{\sqrt{3}}$

$\theta=180^\circ-30^\circ=150^\circ$ $\theta=30^\circ$

Question 3

[Ans: (i) show (ii) $\frac{1}{x-1}-\frac{1}{x+1}-\frac{2}{(x+1)^2}$]

(i) Let $f(x)=x^3+x^2-x-1$

$f(1)=1^3+1^2-1-1=0$

$\therefore x-1$ is a factor of $f(x)$. (shown)

(ii)

$$\begin{array}{r} x^2+2x+1 \\ x-1 \overline{) x^3+x^2-x-1} \\ -(x^3-x^2) \\ \hline 2x^2-x \\ -(2x^2-2x) \\ \hline x-1 \\ -(x-1) \\ \hline 0 \end{array}$$

$\therefore f(x)=(x-1)(x^2+2x+1)=(x-1)(x+1)^2$

Let $\frac{4}{x^3+x^2-x-1}=\frac{4}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$

$4=A(x+1)^2+B(x-1)(x+1)+C(x-1)$

Let $x=1$,

$4=A(1+1)^2+0+0\Rightarrow A=1$

Let $x=-1$,

$4=0+0+C(-1-1)\Rightarrow C=-2$

Let $x=0$,

$4=(1)(1)^2+B(-1)(1)-2(-1)\Rightarrow B=-1$

$\therefore \frac{4}{x^3+x^2-x-1}=\frac{1}{x-1}-\frac{1}{x+1}-\frac{2}{(x+1)^2}$

Question 4

[Ans: (a) $\frac{1}{4}$ (b)(i) $z=\frac{y^2}{1-y}$ (ii) $z>0$; explain]

(a) $\log_2 x+\log_{16} x=-2.5$

$$\log_2 x+\frac{\log_2 x}{\log_2 16}=-2.5$$

$$\log_2 x+\frac{\log_2 x}{\log_2 2^4}=-2.5$$

$$\log_2 x+\frac{\log_2 x}{4}=-2.5$$

$$\frac{5}{4}\log_2 x=-2.5$$

$$\log_2 x=-2 \Rightarrow x=2^{-2}=\frac{1}{4}$$

(b) (i) $\lg z-\lg y=\lg(z+y)$

$$\lg\frac{z}{y}=\lg(z+y)$$

$$\frac{z}{y}=z+y$$

$$z=yz+y^2$$

$$z-yz=y^2 \Rightarrow z=\frac{y^2}{1-y}$$

(ii) For $\lg z$, $z>0$

For $\lg y$, $y>0$

and

$$z=\frac{y^2}{1-y}>0$$

$$\Rightarrow 1-y>0 \Rightarrow y<1$$

$\therefore 0<y<1$

Queston 5

[Ans: show]

$$f''(x) = 3\cos 3x - 4\sin 2x$$

$$f'(x) = \int 3\cos 3x - 4\sin 2x dx$$
$$= 3\left(\frac{\sin 3x}{3}\right) - 4\left(-\frac{\cos 2x}{2}\right) + A$$
$$= \sin 3x + 2\cos 2x + A$$

$$f(x) = \int \sin 3x + 2\cos 2x + A dx$$
$$= -\frac{\cos 3x}{3} + 2\left(\frac{\sin 2x}{2}\right) + Ax + B$$
$$= \sin 2x - \frac{1}{3}\cos 3x + Ax + B$$

$$f(0) = 0$$
$$\sin(0) - \frac{1}{3}\cos(0) + 0 + B = 0 \Rightarrow B = \frac{1}{3}$$

$$f\left(\frac{\pi}{2}\right) = \frac{5}{6}$$
$$\sin\pi - \frac{1}{3}\cos\frac{3\pi}{2} + A\left(\frac{\pi}{2}\right) + \frac{1}{3} = \frac{5}{6}$$
$$0 - \frac{1}{3}(0) + A\left(\frac{\pi}{2}\right) = \frac{1}{2} \Rightarrow A = \frac{1}{\pi}$$

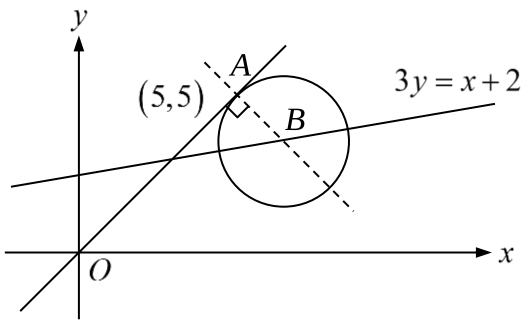
$$f(x) = \sin 2x - \frac{1}{3}\cos 3x + \frac{1}{\pi}x + \frac{1}{3}$$

$$f\left(\frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) - \frac{1}{3}\cos\pi + \frac{1}{\pi}\left(\frac{\pi}{3}\right) + \frac{1}{3}$$
$$= \frac{\sqrt{3}}{2} - \frac{1}{3}(-1) + \frac{1}{3} + \frac{1}{3} = 1 + \frac{\sqrt{3}}{2} \text{ (shown)}$$

Question 6

[Ans: (i) $(x-7)^2+(y-3)^2=8$ (ii) $(7,3-\sqrt{8})$]

(i)



Gradient of $OA=1$

Equation of AB:
$y-5=(-1)(x-5) \Rightarrow y=-x+10$ (1)

$3y=x+2$ (2)

Sub. (1) into (2)
$3(-x+10)=x+2$
$4x=28 \Rightarrow x=7$

Sub. $x=7$ into (1)
$y=-7+10=3$

$\therefore$ centre of the circle is at $(7,3)$.

$AB=\sqrt{(7-5)^2+(3-5)^2}=\sqrt{8}$

Equation of the circle:
$(x-7)^2+(y-3)^2=(\sqrt{8})^2 \Rightarrow (x-7)^2+(y-3)^2=8$

(ii) As the point will be vertically below the centre of the circle,
x-coordinate $=7$
y-coordinate $=3-\sqrt{8}$
$\therefore$ coordinates of the point is $(7,3-\sqrt{8})$.

Question 7

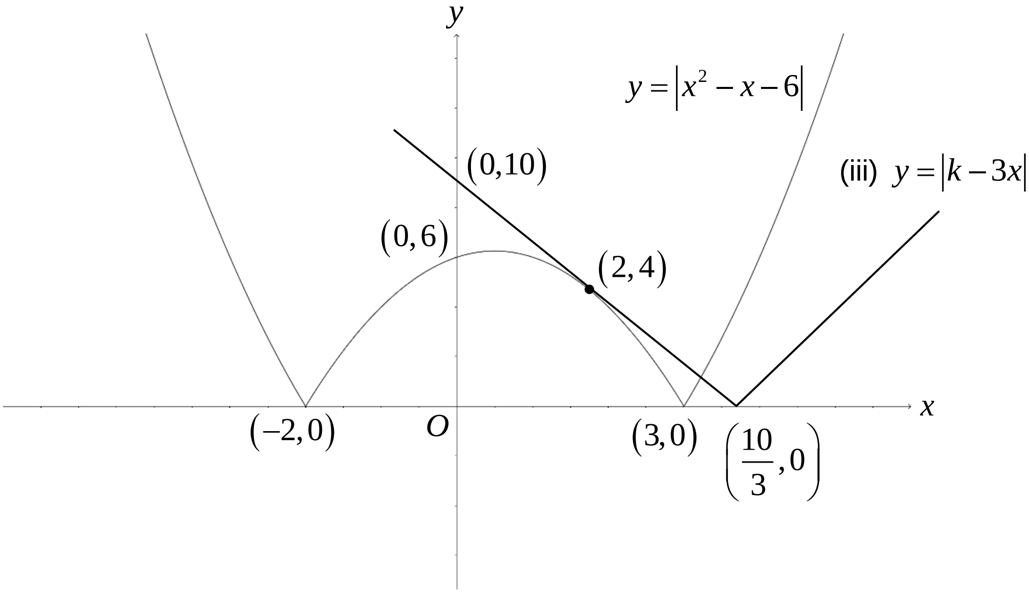
[Ans: (i) sketch (ii) show (iii) sketch]

(i) Let $\left|x^2 - x - 6\right| = 0$

$x^2 - x - 6 = 0$

$(x+2)(x-3) = 0$

$x = -2$ or 3



(ii) When $x = 2$,

$y = \left|2^2 - 2 - 6\right| = 4$

$\therefore (2,4)$ lies on $y = k - 3x$.

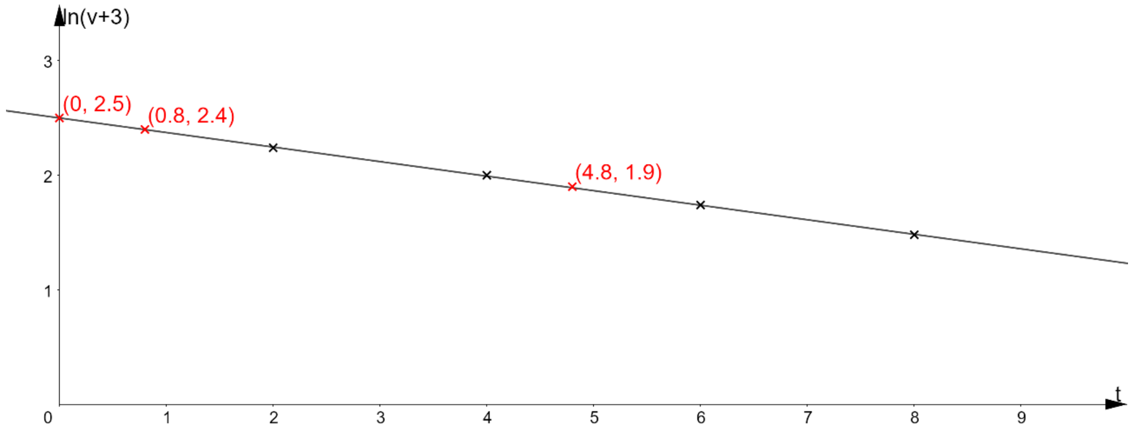
$\Rightarrow k - 3(2) = 4 \Rightarrow k = 10$ (shown)

Question 8

[Ans (i) plot (ii) $A \approx 12.2$, $k \approx -8.00$ (iii) 9.18 m/s (iv) explain (v) 11.2 s]

(i)

t	2	4	6	8
$\ln(v+3)$	2.24	2.00	1.74	1.48



(ii) $v = Ae^{\frac{t}{k}} - 3$

$v + 3 = Ae^{\frac{t}{k}}$

$\ln(v+3) = \ln\left(Ae^{\frac{t}{k}}\right)$

$\ln(v+3) = \frac{1}{k}t + \ln A$

From graph,

Vertical-intercept $= 2.50$

$\ln A = 2.50 \Rightarrow A = e^{2.50} = 12.2$

Gradient $= \frac{2.4 - 1.9}{0.8 - 4.8} = -0.125$

$\frac{1}{k} = -0.125 \Rightarrow k = -8.00$

(iii) When $t = 0$,

$\ln(v+3) = \ln A$

$v + 3 = e^{2.50} \Rightarrow v = 9.18$

(iv) When $v = 5$, $\ln(v+3) = 2.08$. On the graph, t will take up the horizontal axis value when vertical axis value is 2.08.

(v) When $v = 0$,

$\ln(0+3) = -0.125t + 2.50 \Rightarrow t = 11.2$

Question 9

[Ans: $\left(\frac{8}{3}\ln 4-\frac{11}{4}\right)$ units^2]

$$y=\frac{8}{3x+2}=8(3x+2)^{-1}$$

$$\frac{dy}{dx}=8(-1)(3x+2)^{-2}(3)=-\frac{24}{(3x+2)^2}$$

At P,

$$\frac{dy}{dx}=-\frac{24}{\left[3(2)+2\right]^2}=-\frac{3}{8}$$

Equation of BP:

$$y-1=-\frac{3}{8}(x-2)\Rightarrow y=-\frac{3}{8}x+\frac{7}{4}$$

$$\therefore B\left(0,\frac{7}{4}\right)$$

Area of shaded region

$$=\int_0^2\frac{8}{3x+2}\,dx-\frac{1}{2}(2)\left(\frac{7}{4}+1\right)$$

$$=\left[8\left(\frac{\ln|3x+2|}{3}\right)\right]_0^2-\frac{11}{4}$$

$$=\frac{8}{3}\ln 8-\frac{8}{3}\ln 2-\frac{11}{4}$$

$$=\frac{8}{3}\ln 4-\frac{11}{4}$$

Question 10

[Ans: (i) show (ii) $5\beta-12$ (iii) -9 (iv) $4x^2+9x+16=0$]

(i) Since α is a root,

$$\alpha^2 = 3\alpha - 4$$

$$\begin{aligned}\alpha^3 &= \alpha\left(\alpha^2\right)\\ &= \alpha\left(3\alpha-4\right)\\ &= 3\alpha^2-4\alpha\\ &= 3\left(3\alpha-4\right)-4\alpha\\ &= 5\alpha-12 \text{ (shown)}\end{aligned}$$

(ii) Since β is also a root, $\beta^3 = 5\beta - 12$

(iii) For $x^2 = 3x-4 \Rightarrow x^2-3x+4=0$,

$\alpha+\beta=3$ and $\alpha\beta=4$

$$\begin{aligned}\alpha^3+\beta^3 &= \left(5\alpha-12\right)+\left(5\beta-12\right)\\ &= 5\left(\alpha+\beta\right)-24\\ &= 5\left(3\right)-24=-9\end{aligned}$$

(iv) Sum of new roots

$$=\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}$$

$$=-\frac{9}{4}$$

Product of new roots

$$=\left(\frac{\alpha^2}{\beta}\right)\left(\frac{\beta^2}{\alpha}\right)=\alpha\beta=4$$

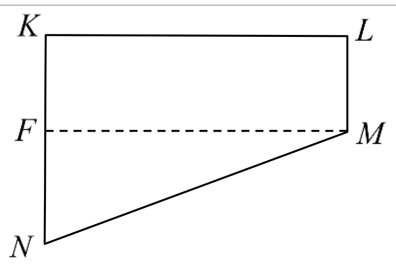
$\therefore$ new equation:

$$x^2-\left(-\frac{9}{4}\right)x+4=0 \Rightarrow 4x^2+9x+16=0$$

Question 11

[Ans: (i) show (ii) $12\sqrt{2}$; $22+12\sqrt{2}$ (iii) show (iv) 1.01]

(i)



$$\sin\theta = \frac{FM}{MN} \Rightarrow FM = MN\sin\theta = 12\sin\theta$$

$$\cos\theta = \frac{FN}{MN} \Rightarrow FN = MN\cos\theta = 12\cos\theta$$

$$P = KL + LM + MN + NF + FK$$
$$= 12\sin\theta + 5 + 12 + 12\cos\theta + 5$$
$$= 22 + 12\cos\theta + 12\sin\theta \text{ (shown)}$$

(ii) Let $12\cos\theta + 12\sin\theta = R\cos(\theta-\alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

By comparison,

$R\cos\alpha = 12$ (1)

$R\sin\alpha = 12$ (2)

$(1)^2+(2)^2$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 12^2 + 12^2$$
$$R^2\left(\cos^2\alpha + \sin^2\alpha\right) = 2\left(12^2\right)$$
$$R^2 = 2\left(12^2\right) \Rightarrow R = 12\sqrt{2}$$

Maximum $P = 22 + 12\sqrt{2}(1) = 22 + 12\sqrt{2}$

(iii) Area, A

$$= \frac{1}{2}(KL)(LM + KN)$$
$$= \frac{1}{2}(12\sin\theta)(5+5+12\cos\theta)$$
$$= 60\sin\theta + 72\sin\theta\cos\theta$$
$$= 60\sin\theta + 36(2\sin\theta\cos\theta)$$
$$= 60\sin\theta + 36\sin 2\theta \text{ (shown)}$$

(iv) $\frac{dA}{d\theta} = 60\cos\theta + 36(2\cos 2\theta) = 60\cos\theta + 72\cos 2\theta$

Let $\frac{dA}{d\theta} = 0$

$$60\cos\theta + 72\cos 2\theta = 0$$
$$5\cos\theta + 6\left(2\cos^2\theta - 1\right) = 0$$
$$12\cos^2\theta + 5\cos\theta - 6 = 0$$
$$\cos\theta = \frac{-5 \pm \sqrt{5^2 - 4(12)(-6)}}{2(12)}$$
$$= \frac{-5+\sqrt{313}}{24} \quad (\because 0 < \theta < \frac{\pi}{2})$$

$\theta = 1.0136$

$$\frac{d^2A}{d\theta^2} = -60\sin\theta + 72\left(-2\sin 2\theta\right) = -60\sin\theta - 144\sin 2\theta$$

When $\theta = 1.0136$,

$$\frac{d^2A}{d\theta^2} = -180.19 < 0$$

$\therefore A$ is a maximum.

When $\theta = 1.01$, the area is at its maximum.