# A-LEVEL H2 MATHS 2018 – PAPER 1

Question 1

[ Ans: (i) $\frac{1-\ln x}{x^2}$ (ii) $1-\frac{2}{e}$ ]

(i) $\frac{dy}{dx}=\frac{x\left(\frac{1}{x}\right)-\ln x}{x^2}=\frac{1-\ln x}{x^2}$

(ii) $\frac{dy}{dx}=\frac{1-\ln x}{x^2}=\frac{1}{x^2}-\frac{\ln x}{x^2}$

$y=\int \frac{1}{x^2}-\frac{\ln x}{x^2}\,dx$

$$\int \frac{\ln x}{x^2}\,dx=\int \frac{1}{x^2}\,dx-y$$
$$=-\frac{1}{x}-y+C$$
$$=-\frac{1}{x}-\frac{\ln x}{x}+C$$

$$\int_1^e \frac{\ln x}{x^2}\,dx=\left[-\frac{1}{x}-\frac{\ln x}{x}\right]_1^e$$
$$=\left(-\frac{1}{e}-\frac{\ln e}{e}\right)-\left(-\frac{1}{1}-\frac{\ln 1}{1}\right)$$
$$=-\frac{1}{e}-\frac{1}{e}+1$$
$$=1-\frac{2}{e}$$

Question 2

[ Ans: (i) $x=\frac{1}{2}$ or $x=3$ (ii) $\frac{125\pi}{6}$ units$^3$ ]

(i) $y=\frac{3}{x}$ (1)

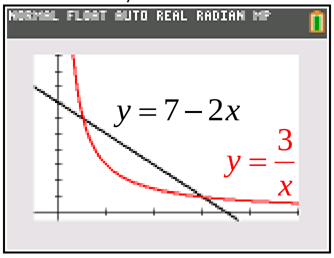
$y+2x=7$ (2)

Sub. (1) into (2)

$$\frac{3}{x}+2x=7$$
$$2x^2-7x+3=0$$
$$(2x-1)(x-3)=0$$
$$x=\frac{1}{2} \text{ or } x=3$$

$\therefore$ $x$-coordinates of $A$ and $B$ are $\frac{1}{2}$ and $3$ respectively.

(ii) From GC,




Volume

$$=\pi\int_{\frac{1}{2}}^{3}(7-2x)^2\,dx-\pi\int_{\frac{1}{2}}^{3}\left(\frac{3}{x}\right)^2\,dx$$
$$=\pi\int_{\frac{1}{2}}^{3}(7-2x)^2\,dx-9\pi\int_{\frac{1}{2}}^{3}x^{-2}dx$$
$$=\pi\left[\frac{(7-2x)^3}{(3)(-2)}\right]_{\frac{1}{2}}^{3}-9\pi\left[\frac{x^{-1}}{-1}\right]_{\frac{1}{2}}^{3}$$
$$=-\frac{\pi}{6}\left[(7-2x)^3\right]_{\frac{1}{2}}^{3}+9\pi\left[\frac{1}{x}\right]_{\frac{1}{2}}^{3}$$
$$=-\frac{\pi}{6}\left[1^3-6^3\right]+9\pi\left(\frac{1}{3}-2\right)$$
$$=\frac{125\pi}{6}$$

Question 3

[ Ans: (i) show; $f\left(x\right)=-\frac{6}{x^3}$ (ii) $y=3-x^2$ ]

(i) $y=ux^2$

$$\frac{dy}{dx}=u\left(2x\right)+\frac{du}{dx}x^2=2ux+x^2\frac{du}{dx}$$

$$x\frac{dy}{dx}=2y-6$$

$$x\left(2ux+x^2\frac{du}{dx}\right)=2ux^2-6$$

$$2ux^2+x^3\frac{du}{dx}=2ux^2-6$$

$$x^3\frac{du}{dx}=-6$$

$\frac{du}{dx}=-\frac{6}{x^3}$ (shown)

$$\Rightarrow f\left(x\right)=-\frac{6}{x^3}$$

(ii) $\frac{du}{dx}=-\frac{6}{x^3}$

$$u=\int -\frac{6}{x^3}\,dx$$

$$=-6\int x^{-3}dx$$

$$=-6\left(\frac{x^{-2}}{-2}\right)+C$$

$$=\frac{3}{x^2}+C$$

$$\therefore \frac{y}{x^2}=\frac{3}{x^2}+C\Rightarrow y=3+Cx^2$$

When $x=1$,

$y=2$

$3+C=2\Rightarrow C=-1$

$$\therefore y=3-x^2$$

Question 4

[ Ans: (i) $x=-1$ or $x=0$ or $x=-1\pm\sqrt{3}$ (ii) $-1-\sqrt{3}<x<-1$ or $0<x<-1+\sqrt{3}$ ]

(i) $\left|2x^2+3x-2\right|=2-x$

$2x^2+3x-2=-(2-x)$ or $2x^2+3x-2=2-x$

$2x^2+2x=0$

$x(x+1)=0$

$x=-1$ or $x=0$

$2x^2+4x-4=0$

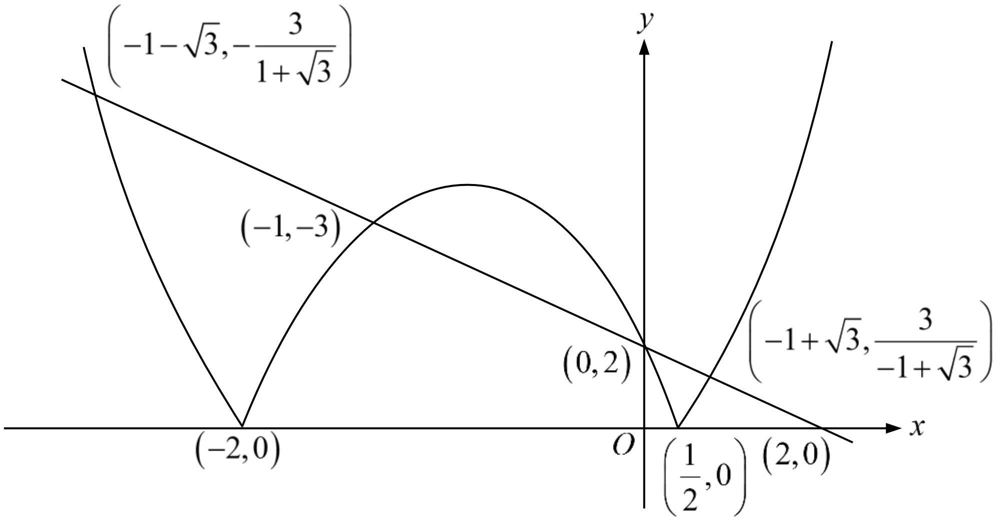
$x^2+2x-2=0$

$(x+1)^2-3=0$

$(x+1)^2=3$

$x=-1\pm\sqrt{3}$

(ii)




$\left|2x^2+3x-2\right|<2-x$

$-1-\sqrt{3}<x<-1$ or $0<x<-1+\sqrt{3}$

Queston 5

[ Ans: $b=-1$; $f^{-1}(x)=\dfrac{x+a}{x-1}$ ]

$$ff(x)=\frac{\dfrac{x+a}{x+b}+a}{\dfrac{x+a}{x+b}+b}$$

$$=\frac{x+a+a(x+b)}{x+a+b(x+b)}$$

$$=\frac{(a+1)x+a+ab}{(b+1)x+a+b^2}$$

$$ff(x)=g(x)$$

$$\frac{(a+1)x+a+ab}{(b+1)x+a+b^2}=x$$

From observation, since there should be no $x$ in the denominator,

$$b+1=0\Rightarrow b=-1$$

$$ff(x)=g(x)$$

$$ff(x)=x$$

$$f^{-1}\left(ff(x)\right)=f^{-1}(x)$$

$$f(x)=f^{-1}(x)$$

$$f^{-1}(x)=\frac{x+a}{x+b}=\frac{x+a}{x-1}$$

Question 6

[ Ans: (i) show (ii) $\pm 2\sqrt{31}$ ]

(i) $\underset{\sim}{a} \times 3\underset{\sim}{b} = 2\underset{\sim}{a} \times \underset{\sim}{c}$

$\underset{\sim}{a} \times 3\underset{\sim}{b} - 2\underset{\sim}{a} \times \underset{\sim}{c} = \underset{\sim}{0}$

$\underset{\sim}{a} \times \left(3\underset{\sim}{b} - 2\underset{\sim}{c}\right) = \underset{\sim}{0}$

$\therefore \underset{\sim}{a}$ is parallel to $3\underset{\sim}{b} - 2\underset{\sim}{c}$.

$3\underset{\sim}{b} - 2\underset{\sim}{c} = \lambda \underset{\sim}{a}$ (shown)

(ii) $\left(3\underset{\sim}{b} - 2\underset{\sim}{c}\right) \cdot \left(3\underset{\sim}{b} - 2\underset{\sim}{c}\right) = \left(\lambda \underset{\sim}{a}\right) \cdot \left(\lambda \underset{\sim}{a}\right)$

$9\underset{\sim}{b} \cdot \underset{\sim}{b} - 6\underset{\sim}{b} \cdot \underset{\sim}{c} - 6\underset{\sim}{c} \cdot \underset{\sim}{b} + 4\underset{\sim}{c} \cdot \underset{\sim}{c} = \left|\lambda \underset{\sim}{a}\right|^2$

$9\left|\underset{\sim}{b}\right|^2 - 12\underset{\sim}{b} \cdot \underset{\sim}{c} + 4\left|\underset{\sim}{c}\right|^2 = \left|\lambda\right|^2 \left|\underset{\sim}{a}\right|^2$

$9\left|\underset{\sim}{b}\right|^2 - 12\left|\underset{\sim}{b}\right|\left|\underset{\sim}{c}\right| \cos 60° + 4\left|\underset{\sim}{c}\right|^2 = \lambda^2 \left|\underset{\sim}{a}\right|^2$

$9(4)^2 - 12(4)(1)\left(\frac{1}{2}\right) + 4(1)^2 = \lambda^2 (1)^2$

$\lambda^2 = 124$

$\lambda = \pm\sqrt{124} = \pm 2\sqrt{31}$

Question 7

[ Ans: (i) $\frac{dy}{dx}=\frac{2x-y^2}{2xy+16y}$ (ii) $N\left(-\frac{1}{17},0\right)$ ]

(i) $\frac{x^2-4y^2}{x^2+xy^2}=\frac{1}{2}$

$2x^2-8y^2=x^2+xy^2$

$x^2-8y^2=xy^2$

$\frac{d}{dx}\left(x^2-8y^2\right)=\frac{d}{dx}\left(xy^2\right)$

$2x-16y\frac{dy}{dx}=2xy\frac{dy}{dx}+y^2$

$\left(2xy+16y\right)\frac{dy}{dx}=2x-y^2$

$\frac{dy}{dx}=\frac{2x-y^2}{2xy+16y}$

(ii) When $x=1$ for curve $C$,

$1^2-8y^2=(1)y^2$

$y^2=\frac{1}{9}\Rightarrow y=\pm\frac{1}{3}$

Let coordinates of $P$ and $Q$ be $\left(1,-\frac{1}{3}\right)$ and $\left(1,\frac{1}{3}\right)$ respectively.

At $P$,

$$\frac{dy}{dx}=\frac{2(1)-\left(-\frac{1}{3}\right)^2}{2(1)\left(-\frac{1}{3}\right)+16\left(-\frac{1}{3}\right)}=-\frac{17}{54}$$

Equation of tangent: $y-\left(-\frac{1}{3}\right)=-\frac{17}{54}(x-1)\Rightarrow y=-\frac{17}{54}x-\frac{1}{54}$ (1)

At $Q$,

$$\frac{dy}{dx}=\frac{2(1)-\left(\frac{1}{3}\right)^2}{2(1)\left(\frac{1}{3}\right)+16\left(\frac{1}{3}\right)}=\frac{17}{54}$$

Equation of tangent: $y-\frac{1}{3}=\frac{17}{54}(x-1)\Rightarrow y=\frac{17}{54}x+\frac{1}{54}$ (2)

Solving (1) and (2) using GC, $x=-\frac{1}{17}$, $y=0$

$\therefore N\left(-\frac{1}{17},0\right)$

Question 8

[ Ans: (i) $A=5$; $u_3=40$ (ii) $a=\frac{15}{2}$, $b=-5$, $c=-5$ (iii) $15\left(2^n-1\right)-\frac{5}{2}n\left(n+1\right)-5n$ ]

(i) Given $u_{n+1}=2u_n+An$, where $n\geq 1$

For $n=1$,

$$u_2=2u_1+A\left(1\right)$$

$$A=u_2-2u_1=15-2\left(5\right)=5$$

$$u_{n+1}=2u_n+5n$$

For $n=2$,

$$u_3=2u_2+5\left(2\right)=2\left(15\right)+10=40$$

(ii) Given $u_n=a\left(2^n\right)+bn+c$

$u_1=5$

$a\left(2^1\right)+b\left(1\right)+c=5\Rightarrow 2a+b+c=5$ (1)

$u_2=15$

$a\left(2^2\right)+b\left(2\right)+c=15\Rightarrow 4a+2b+c=15$ (2)

$u_3=40$

$a\left(2^3\right)+b\left(3\right)+c=40\Rightarrow 8a+3b+c=5$ (3)

From GC,

$$a=\frac{15}{2},\ b=-5,\ c=-5$$

(iii) $u_n=\frac{15}{2}\left(2^n\right)-5n-5$

$$\sum_{r=1}^{n}u_r$$

$$=\sum_{r=1}^{n}\left[\frac{15}{2}\left(2^r\right)-5r-5\right]=\frac{15}{2}\sum_{r=1}^{n}2^r-5\sum_{r=1}^{n}r-\sum_{r=1}^{n}5$$

$$=\frac{15}{2}\left(2+2^2+2^3+\ldots 2^n\right)-5\left[\frac{n\left(n+1\right)}{2}\right]-\left(n-1+1\right)\left(5\right)$$

$$=\frac{15}{2}\left[\frac{2\left(2^n-1\right)}{2-1}\right]-\frac{5}{2}n\left(n+1\right)-5n$$

$$=15\left(2^n-1\right)-\frac{5}{2}n\left(n+1\right)-5n$$

Question 9

[ Ans: (i) show (ii) $k=2$ (iii) $8$ units ]

(i) $\dfrac{dx}{d\theta}=2-2\cos 2\theta=2\left(1-\cos 2\theta\right)=2\left(2\sin^2\theta\right)=4\sin^2\theta$ ; $\dfrac{dy}{d\theta}=4\sin\theta\cos\theta$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$$

$$=\frac{4\sin\theta\cos\theta}{4\sin^2\theta}$$

$$=\frac{\cos\theta}{\sin\theta}=\cot\theta \text{ (shown)}$$

(ii) When $\theta=\alpha$ ,

$$x=2\alpha-\sin 2\alpha$$

$$y=2\sin^2\alpha$$

$$\frac{dy}{dx}=\cot\alpha$$

Equation of normal:

$$y-\left(2\sin^2\alpha\right)=-\frac{1}{\cot\alpha}\left[x-\left(2\alpha-\sin 2\alpha\right)\right]$$

At $A$,

$$0-\left(2\sin^2\alpha\right)=-\frac{1}{\cot\alpha}\left[x-\left(2\alpha-\sin 2\alpha\right)\right]$$

$$2\sin^2\alpha\cot\alpha=x-\left(2\alpha-\sin 2\alpha\right)$$

$$x=2\sin\alpha\cos\alpha+\left(2\alpha-\sin 2\alpha\right)$$

$$=\sin 2\alpha+2\alpha-\sin 2\alpha$$

$$=2\alpha \text{ (shown)}$$

$$k=2$$

(iii) To obtain the total length of the arc, let $\beta=0$ , $\gamma=\pi$

Total length

$$=\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$=\int_0^{\pi}\sqrt{\left(4\sin^2\theta\right)^2+\left(4\sin\theta\cos\theta\right)^2}\,d\theta$$

$$=\int_0^{\pi}\sqrt{16\sin^4\theta+16\sin^2\theta\cos^2\theta}\,d\theta$$

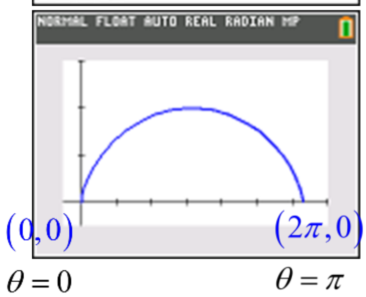
$$=\int_0^{\pi}\sqrt{16\sin^2\theta\left(\sin^2\theta+\cos^2\theta\right)}\,d\theta$$

$$=\int_0^{\pi}4\sin\theta\,d\theta$$

$$=4\left[-\cos\theta\right]_0^{\pi}$$

$$=4\left(-\left(\cos\pi\right)+\cos 0\right)=4\left[-\left(-1\right)+1\right]=8$$

From GC,

Question 10

[ Ans: (i) show; $V$ is constant (ii) show (iii) max. $I=\frac{3A}{2e}$; show maximum (iv) sketch ]

(i) $L\frac{dI}{dt}+RI+\frac{q}{C}=V$

$$\frac{d}{dt}\left(L\frac{dI}{dt}+RI+\frac{q}{C}\right)=\frac{d}{dt}(V)$$

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}\frac{dq}{dt}=\frac{dV}{dt}$$

$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{I}{C}=0$, for $I=\frac{dq}{dt}$ and $V$ is a constant

(ii) $I=Ate^{-\frac{Rt}{2L}}$

$$\frac{dI}{dt}=Ate^{-\frac{Rt}{2L}}\left(-\frac{R}{2L}\right)+Ae^{-\frac{Rt}{2L}}$$

$$=-\frac{R}{2L}\left(Ate^{-\frac{Rt}{2L}}\right)+Ae^{-\frac{Rt}{2L}}$$

$$=-\frac{R}{2L}I+\frac{I}{t}=\left(\frac{1}{t}-\frac{R}{2L}\right)I$$

$$\frac{d^2I}{dt^2}=\left(\frac{1}{t}-\frac{R}{2L}\right)\frac{dI}{dt}-\frac{1}{t^2}I$$

$$L\frac{d^2I}{dt^2}=\left(\frac{L}{t}-\frac{R}{2}\right)\frac{dI}{dt}-\frac{L}{t^2}I$$

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}=\left(\frac{L}{t}-\frac{R}{2}\right)\frac{dI}{dt}-\frac{L}{t^2}I+R\frac{dI}{dt}$$

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}=\left(\frac{L}{t}+\frac{R}{2}\right)\frac{dI}{dt}-\frac{L}{t^2}I$$

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}=\left(\frac{L}{t}+\frac{R}{2}\right)\left(\frac{1}{t}-\frac{R}{2L}\right)I-\frac{L}{t^2}I$$

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}=\left(\frac{L}{t^2}-\frac{R}{2t}+\frac{R}{2t}-\frac{R^2}{4L}\right)I-\frac{L}{t^2}I$$

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}=-\frac{R^2}{4L}I$$

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{R^2}{4L}I=0$$

$\therefore\frac{1}{C}=\frac{R^2}{4L}\Rightarrow C=\frac{4L}{R^2}$ (shown)

(iii) Let $\frac{dI}{dt}=0$

$$\left(\frac{1}{t}-\frac{R}{2L}\right)I=0$$

$$\frac{1}{t}-\frac{R}{2L}=0$$ ($\because I \neq 0$ at maximum since $I \geq 0$)

$$t=\frac{2L}{R}=\frac{2(3)}{4}=\frac{3}{2}$$

When $t=\frac{3}{2}$,

$$L\frac{d^2I}{dt^2}+R(0)+\frac{R^2}{4L}I=0$$

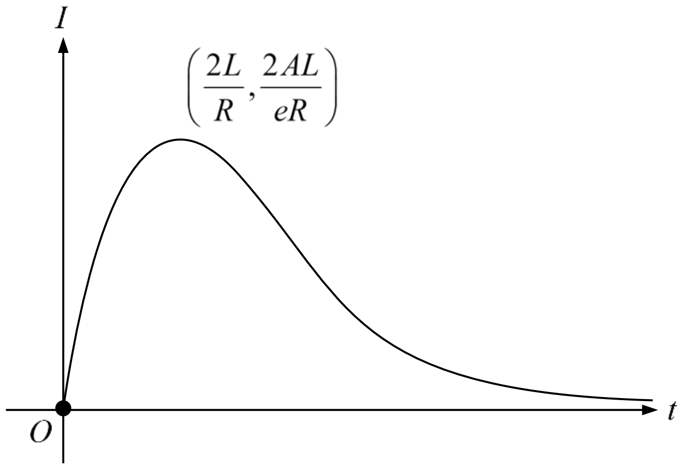
$$\frac{d^2I}{dt^2}=-\frac{R^2}{4L^2}I=-\frac{R^2}{4L^2}I<0$$ ($\because I=Ate^{-\frac{Rt}{2L}}>0$ for $t \neq 0$)

$\therefore I$ is a maximum

Max. $I=A\left(\frac{3}{2}\right)e^{-\frac{4}{2(3)}\left(\frac{3}{2}\right)}=\frac{3A}{2e}$

(iv) When $t=\frac{2L}{R}$,

$$I=A\left(\frac{2L}{R}\right)e^{-\frac{R}{2L}\left(\frac{2L}{R}\right)}=\frac{2AL}{eR}$$



Question 11

[ Ans: (i)(a) $\$102.43$ (b) $\$1215.71$ (c) June $2018$; beginning of month

(ii)(a) $100+12b$ (b) $b=\frac{4}{3}$ (iii) $b=1.23$ ]

(i) (a) Worth of $\$100$ invested on $1$ Jan 2016 by $31$ Dec 2016

$$=100\left(1+\frac{a}{100}\right)^{12}=100\left(1+\frac{0.2}{100}\right)^{12}=100\left(1.002\right)^{12}$$

$$=\$102.43$$

(b)

| Month | Total amount in the account at end of month |
|---|---|
| 1 (Jan) | $100\left(1.002\right)$ |
| 2 (Feb) | $\left(100+100\left(1.002\right)\right)\left(1.002\right)$<br>$=100\left(1.002\right)+100\left(1.002\right)^{2}$ |
| 3 (Mar) | $\left(100+100\left(1.002\right)+100\left(1.002\right)^{2}\right)\left(1.002\right)$<br>$=100\left(1.002\right)+100\left(1.002\right)^{2}+100\left(1.002\right)^{3}$ |
| ⋮ | |
| 12 (Dec) | $\frac{100\left(1.002\right)\left(1.002^{12}-1\right)}{1.002-1}$<br>$=\$1215.71$ |

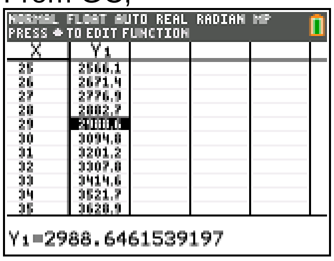
(c) Total amount in the account at the end of $n$ th month

$$\frac{100\left(1.002\right)\left(1.002^{n}-1\right)}{1.002-1}$$

$$=50100\left(1.002^{n}-1\right)$$

Let $50100\left(1.002^{n}-1\right)<3000$

From GC,




Maximum $n=29$

Total amount in the account at the beginning of $30$ th month

$=2988.6+100=3088.6>3000$

$\therefore$ The total amount in the account will first exceed $\$3000$ in June $2018$, and it will occur in the beginning of the month.

(ii) (a) Worth of $\$100$ invested on $1$ Jan 2016 by $31$ Dec 2016
$=100+12b$

(b)

| Month | Total amount in the account at end of month |
|---|---|
| $1$ (Jan) | $100+b$ |
| $2$ (Feb) | $(100+2b)+(100+b)$<br>$=2(100)+(b+2b)$ |
| $3$ (Mar) | $(100+3b)+(100+2b)+(100+b)$<br>$=3(100)+(b+2b+3b)$ |
| $\vdots$ | |
| $24$ | $24(100)+\frac{24}{2}(b+24b)$<br>$=2400+300b$ |

Let $2400+300b=2800$
$300b=400$
$b=\frac{4}{3}$

(iii) Under plan $P$, total amount in the account at the end of $60$ th month
$$=\frac{100(1.01)(1.01^{60}-1)}{1.01-1}$$
$$=10100(1.01^{60}-1)$$

Under plan $Q$, total amount in the account at the end of $60$ th month
$$=60(100)+\frac{60}{2}(b+60b)$$
$$=6000+1830b$$

Let $10100(1.01^{60}-1)=6000+1830b$
$$b=\frac{10100(1.01^{60}-1)-6000}{1830}$$
$$=1.23$$