O-LEVEL A-MATHS 2018 – PAPER 2

Question 1

[Ans: (i) show (ii) $3x^2+2x+4=0$]

(i) Sum of roots $=-3 \Rightarrow \alpha+\beta=-3$

Product of roots $=5 \Rightarrow \alpha\beta=5$

$(\alpha+1)(\beta+1)$

$=\alpha\beta+\alpha+\beta+1$

$=5+(-3)+1=3$ (shown)

(ii) For the new quadratic equation,

Sum of roots

$=\dfrac{2}{\alpha+1}+\dfrac{2}{\beta+1}$

$=\dfrac{2\beta+2+2\alpha+2}{(\alpha+1)(\beta+1)}$

$=\dfrac{2(\alpha+\beta)+4}{(\alpha+1)(\beta+1)}$

$=\dfrac{2(-3)+4}{3}=-\dfrac{2}{3}$

Product of roots

$=\left(\dfrac{2}{\alpha+1}\right)\left(\dfrac{2}{\beta+1}\right)$

$=\dfrac{4}{(\alpha+1)(\beta+1)}$

$=\dfrac{4}{3}$

New quadratic equation:

$x^2-\left(-\dfrac{2}{3}\right)x+\left(\dfrac{4}{3}\right)=0$

$3x^2+2x+4=0$

Question 2

[Ans: $a=\frac{1}{2}$; $b=-324$]

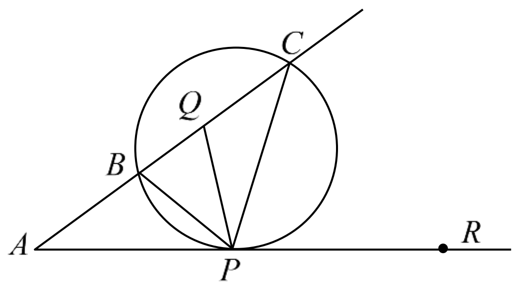
$(1-4x)(2+ax)^6$
$=(1-4x)\left[2^6+\binom{6}{1}(2)^5(ax)+\binom{6}{2}(2)^4(ax)^2+\ldots\right]$
$=(1-4x)(64+192ax+240a^2x^2+\ldots)$
$=64+192ax+240a^2x^2-256x-768ax^2+\ldots$
$=64+(192a-256)x+(240a^2-768a)x^2+\ldots$

$\therefore 192a-256=-160 \Rightarrow a=\frac{1}{2}$

$\therefore b=240\left(\frac{1}{2}\right)^2-768\left(\frac{1}{2}\right)=-324$

Question 3

[Ans: prove]



$\angle CPQ=180°-\angle CPR-\angle APQ$ ($\angle$ sum on a straight line)

$\angle BPQ$
$=180°-\angle QBP-\angle BQP$ ($\angle$ sum of Δ)
$=180°-\angle CPR-\angle BQP$ (alternate segment theorem)
$=180°-\angle CPR-\angle APQ$ (base $\angle$s of isosceles Δ)
$=\angle CPQ$

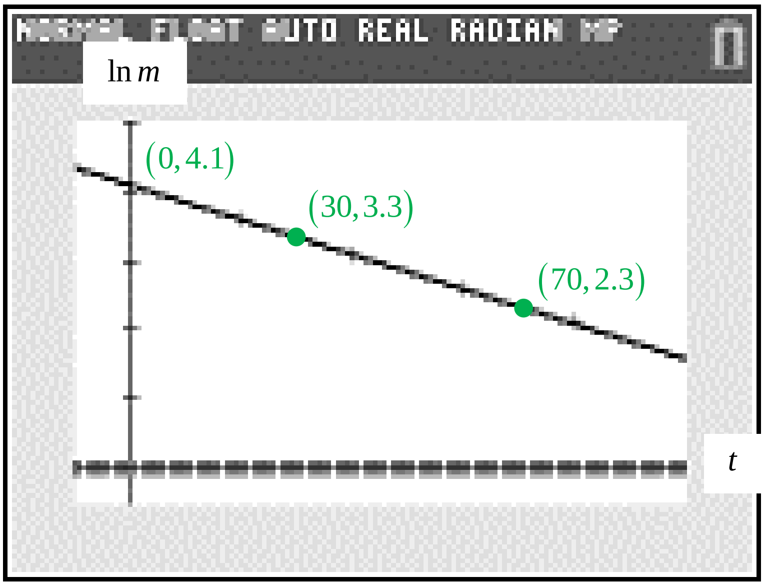
$\therefore PQ$ bisects $\angle BPC$ (proven)

Question 4

[Ans: (i) plot (ii) 60.3 (iii) 0.025 (iv) 28 h]

(i)

$\ln m$	3.6	3.1	2.6	2.1
t	20	40	60	80



(ii) From graph, when $t = 0$,

$\ln m = 4.1$

$m = e^{4.1} = 60.3$

(iii) $m = m_0 e^{-kt}$

$\ln m = \ln m_0 - kt$

$\therefore -k =$ gradient of graph

$$-k = \frac{3.3 - 2.3}{30 - 70}$$

$k = 0.025$

(iv) When $t = 0$, $m = m_0 e^{-k(0)} = m_0$

When $m = \frac{1}{2} m_0$,

$$m_0 e^{-0.025t} = \frac{1}{2} m_0$$

$$e^{-0.025t} = \frac{1}{2}$$

$$-0.025t = \ln \frac{1}{2} \Rightarrow t = 27.726$$

It takes approximately 28 h for the substance to lose half of its original mass.

Queston 5

[Ans: (i) $BC = 800\tan\theta$; $CD = 1200 - 800\tan\theta$ (ii) show (iii) $48.3°$]

(i) $\tan\theta = \dfrac{BC}{AB}$

$BC = AB\tan\theta = 800\tan\theta$

$CD = 1200 - 800\tan\theta$

(ii) $\angle CDE = \angle BAC = \theta$

$\cos\theta = \dfrac{DE}{CD}$

$DE = CD\cos\theta = (1200 - 800\tan\theta)\cos\theta$

$= \left(1200 - 800\dfrac{\sin\theta}{\cos\theta}\right)\cos\theta$

$= 1200\cos\theta - 800\sin\theta$ (shown)

(iii) Let $DE = 1200\cos\theta - 800\sin\theta = R\cos(\theta + \alpha)$

$1200\cos\theta - 800\sin\theta = R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$

$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$\therefore R\cos\alpha = 1200$ (A) $R\sin\alpha = 800$ (B)

$(A)^2 + (B)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 1200^2 + 800^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 1200^2 + 800^2$

$R^2 = 1200^2 + 800^2 \Rightarrow R = \sqrt{2080000} = 400\sqrt{13}$

$\dfrac{(B)}{(A)}$

$\dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{800}{1200}$

$\tan\alpha = \dfrac{2}{3} \Rightarrow \alpha = \tan^{-1}\dfrac{2}{3} = 33.690$

$\therefore DE = 400\sqrt{13}\cos(\theta + 33.690°)$

$DE = 200$

$400\sqrt{13}\cos(\theta + 33.690°) = 200$

$\cos(\theta + 33.690°) = \dfrac{200}{400\sqrt{13}}$

$\theta = \cos^{-1}\dfrac{200}{400\sqrt{13}} - 33.690°$

$= 48.3°$

Question 6

[Ans: (i) $\cos x - x\sin x$ (ii) $\sin x - x\cos x + c$ (iii) $x^2\cos x + 2x\sin x$;
$x^2\sin x + 2\left(x\cos x - \sin x\right) + c$]

(i) $\frac{d}{dx}\left(x\cos x\right) = x\left(-\sin x\right) + \left(1\right)\cos x = \cos x - x\sin x$

(ii) $\int \frac{d}{dx}\left(x\cos x\right)dx = \int \cos x - x\sin x dx$

$x\cos x = \int \cos x dx - \int x\sin x dx$

$\int x\sin x dx = \int \cos x dx - x\cos x$

$= \sin x - x\cos x + c$

(iii) $\frac{d}{dx}\left(x^2\sin x\right) = x^2\left(\cos x\right) + \left(2x\right)\sin x = x^2\cos x + 2x\sin x$

$\int \frac{d}{dx}\left(x^2\sin x\right)dx = \int x^2\cos x + 2x\sin x dx$

$x^2\sin x = \int x^2\cos x dx + 2\int x\sin x dx$

$\int x^2\cos x dx = x^2\sin x - 2\int x\sin x dx$

$= x^2\sin x - 2\left(\sin x - x\cos x\right) + c$

$= x^2\sin x + 2\left(x\cos x - \sin x\right) + c$

Question 7

[Ans: (i) 7.27 m/s; -0.116 m/s^2 (ii) explain (iii) 116 m]

(i) v

$$= \frac{d}{dt}\left[840\left(1-e^{-\frac{t}{80}}\right)-2t\right]$$

$$= 840\left(-e^{-\frac{t}{80}}\right)\left(-\frac{1}{80}\right)-2$$

$$= \frac{21}{2}e^{-\frac{t}{80}}-2$$

a

$$= \frac{d}{dt}\left(\frac{21}{2}e^{-\frac{t}{80}}-2\right)$$

$$= \frac{21}{2}e^{-\frac{t}{80}}\left(-\frac{1}{80}\right)$$

$$= -\frac{21}{160}e^{-\frac{t}{80}}$$

When $t=10$,

$$v = \frac{21}{2}e^{-\frac{10}{80}}-2 = 7.27$$

$$a = -\frac{21}{160}e^{-\frac{10}{80}} = -0.116$$

10 s after the girl passes A, her speed is 7.27 m/s and acceleration is -0.116 m/s^2.

(ii) The sign of acceleration is negative, which indicates that the girl's speed is decreasing as time increases.

(iii) When $v=1.5$,

$$\frac{21}{2}e^{-\frac{t}{80}}-2=1.5$$

$$e^{-\frac{t}{80}}=\frac{1}{3}$$

$$-\frac{t}{80}=\ln\frac{1}{3} \Rightarrow t=-80\ln\frac{1}{3}$$

Distance the girl has to push her bicycle

$$= 500-\left[840\left(1-e^{-\frac{-80\ln\frac{1}{3}}{80}}\right)-2\left(-80\ln\frac{1}{3}\right)\right]$$

$$= 116$$

Question 8

[Ans: (i) -14 (ii) show (iii) show (iv) 0.585]

(i) Remainder

$$= p(-2) = 2(-2)^3 + 5(-2)^2 - 18 = -14$$

(ii) $p\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 + 5\left(\frac{3}{2}\right)^2 - 18 = 0$

$\therefore 2x-3$ is a factor of $p(x)$.

(iii)

$$\begin{array}{r}
x^2+4x+6 \\
2x-3\overline{\big)\,2x^3+5x^2+0x-18} \\
-\left(2x^3-3x^2\right) \\
\hline
8x^2+0x \\
-\left(8x^2-12x\right) \\
\hline
12x-18 \\
-\left(12x-18\right) \\
\hline
0
\end{array}$$

$\therefore p(x) = 0$

$(2x-3)(x^2+4x+6) = 0$

$2x-3=0$ or $x^2+4x+6=0$

$x = \frac{3}{2}$

Discriminant

$= 4^2 - 4(1)(6) = -8 < 0$

$\therefore$ no real roots

$\therefore p(x) = 0$ has only one real root.

(iv) $2^{3y+1} + 5\left(2^{2y}\right) = 18$

$2^{3y}(2) + 5\left(2^{2y}\right) = 18$

$2\left(2^y\right)^3 + 5\left(2^y\right)^2 = 18$

$2^y = \frac{3}{2}$

$y \ln 2 = \ln\frac{3}{2} \Rightarrow y = 0.585$

Question 9

[Ans: (i) show; $(3, 44)$ (ii) explain; $k = 2$]

(i) When $k = 5$,

$y = 2x^2 + (5+2)x + 5 \Rightarrow y = 2x^2 + 7x + 5$ (1)

$y = 19x - 13$ (2)

(1) = (2)

$2x^2 + 7x + 5 = 19x - 13$

$2x^2 - 12x + 18 = 0$

$x^2 - 6x + 9 = 0$

Discriminant

$= (-6)^2 - 4(1)(9) = 0$

$\therefore$ the line $y = 19x - 13$ is a tangent to the curve. (shown)

$x^2 - 6x + 9 = 0$

$(x-3)^2 = 0 \Rightarrow x = 3$

Sub. $x = 3$ into (2)

$y = 19(3) - 13 = 44$

$\therefore$ coordinates of the point of contact is $(3, 44)$.

(ii) As the coefficient of x^2 in the equation $y = 2x^2 + (k+2)x + k$ is positive, for y to be not negative,

Discriminant ≤ 0

$(k+2)^2 - 4(2)(k) \leq 0$

$k^2 + 4k + 4 - 8k \leq 0$

$k^2 - 4k + 4 \leq 0$

$(k-2)^2 \leq 0$

$(k-2)^2 < 0$ (NA) or $(k-2)^2 = 0$, $k = 2$

$\therefore$ there is only one value of k for which y cannot be negative, and this value of k is 2.

Question 10

[Ans: (i) prove (ii) show (iii) $\frac{16}{9}$ units²]

(i) $y = 2\sqrt{7-3x} = 2\left(7-3x\right)^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\left(\frac{1}{2}\right)\left(7-3x\right)^{-\frac{1}{2}}\left(-3\right) = -\frac{3}{\sqrt{7-3x}}$

At P,

$\frac{dy}{dx} = -\frac{3}{\sqrt{7-3k}}$

Equation of normal: $y - 2\sqrt{7-3k} = \frac{\sqrt{7-3k}}{3}\left(x-k\right)$

Since this normal passes through $\left(-5,0\right)$,

$0 - 2\sqrt{7-3k} = \frac{\sqrt{7-3k}}{3}\left(-5-k\right)$

$-2\sqrt{7-3k} = -\frac{\sqrt{7-3k}}{3}\left(5+k\right)$

$2 = \frac{1}{3}\left(5+k\right) \Rightarrow 5+k = 6 \Rightarrow k = 1$

(ii) For $k = 1$,

$P\left(1, 2\sqrt{7-3}\right) = P\left(1,4\right)$

Gradient at $P = -\frac{3}{\sqrt{7-3}} = -\frac{3}{2}$

Equation of tangent at P: $y - 4 = -\frac{3}{2}\left(x-1\right) \Rightarrow y = -\frac{3}{2}x + \frac{11}{2}$

At T, $y = 0$

$-\frac{3}{2}x + \frac{11}{2} = 0$

$\frac{3}{2}x = \frac{11}{2} \Rightarrow x = \frac{11}{3}$ (shown)

(iii) Area

$= \frac{1}{2}\left(\frac{11}{3} - 1\right)\left(4\right) - \int_1^{\frac{7}{3}} 2\left(7-3x\right)^{\frac{1}{2}}\, dx$

$= \frac{16}{3} - 2\left[\frac{\left(7-3x\right)^{\frac{3}{2}}}{\left(-3\right)\left(\frac{3}{2}\right)}\right]_1^{\frac{7}{3}} = \frac{16}{3} - \frac{4}{9}\left[\left(7-3x\right)^{\frac{3}{2}}\right]_1^{\frac{7}{3}}$

$= \frac{16}{3} - \frac{4}{9}\left[\left(0\right)^{\frac{3}{2}} - \left(4\right)^{\frac{3}{2}}\right] = \frac{16}{3} - \frac{32}{9} = \frac{16}{9}$

Question 11

[Ans: (i) show (ii) explain (iii) $(x-4)^2+(y-8)^2=25$ (iv) explain (v) $y=-\frac{3}{4}x+\frac{69}{4}$]

(i) Gradient of $AB=\frac{8-4}{9-1}=\frac{1}{2}$

Gradient of $BC=\frac{12-8}{7-9}=-2$

$\left(\text{Gradient of } AB\right)\left(\text{Gradient of } BC\right)=\left(\frac{1}{2}\right)(-2)=-1$

$\therefore \angle ABC=90°$

(ii) Since $\angle ABC=90°$, $\therefore$ A, B and C lie on a circle with diameter AC. ($\angle$ in a semicircle)

(iii) Center of circle $=$ Midpoint of AC

$=\left(\frac{1+7}{2},\frac{4+12}{2}\right)=(4,8)$

Length of diameter $=AC=\sqrt{(1-7)^2+(4-12)^2}=10$

Equation of circle:

$(x-4)^2+(y-8)^2=\left(\frac{10}{2}\right)^2$

$(x-4)^2+(y-8)^2=25$

(iv) y-coordintes of both B and the centre of the circle are 8, $\therefore$ they are parallel to the x-axis. Since tangent to the circle at B is perpendicular to the radius, $\therefore$ this tangent is parallel to the y-axis.

(v) Let the centre of the circle be D. $\therefore D(4,8)$

Gradient of $CD=\frac{12-8}{7-4}=\frac{4}{3}$

Equation of tangent to the circle at C:

$y-12=-\frac{3}{4}(x-7)$

$y=-\frac{3}{4}x+\frac{69}{4}$