ST ANDREW'S JUNIOR COLLEGE

COMMON TEST

MATHEMATICS
Higher 2 **9758**

Tuesday **27 June 2017** **2 h 30 min**

Additional materials : Answer paper
List of Formulae (MF26)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.

Answer **all** the questions. Total marks : **80**

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of **7** printed pages including this page.

[Turn over

1 Four friends went to a fruit seller to buy durians. The weight of the different varieties of durian and the total amount paid by each person are shown in the table below:

	Adam	Bryan	Charlie	Wayne
D13 (kg)	4.2	3.7	3.3	2.9
Golden Phoenix (kg)	3.1	0	3.7	4.2
Musang King (kg)	1.5	4.8	3.0	1.2
Total amount paid ($)	98.60	118.20	131.60	k

Assuming that, for each variety of durian, the price per kilogram paid by each of the friends is the same, calculate the total amount, $ k, paid by Wayne. [4]

2 The sum of the first n terms of a series is given by $S_n = 243 - 3^{5-n}$. Find the nth term of the series and prove that it is a geometric series. Show that this series is convergent and find its sum to infinity. [4]

3 **(a)** Given that the graphs of $y = \mathrm{f}(mx)$ and $y = \mathrm{f}(x)$ are identical for $x \in \mathbb{R}$, where m is any non-zero real number, state a possible equation of f(x). [1]

(b) A graph with equation $y = \mathrm{f}(x)$ undergoes a sequence of transformations ***A, B, C*** and ***D*** in this order, where ***A***, ***B***, ***C*** and ***D*** are defined as follows:

A: A translation of 2 units in the negative direction of the x-axis

B: A scaling parallel to the x-axis by a factor $\frac{1}{3}$

C: A reflection in the x-axis

D: A translation of 5 units in the positive direction of the y-axis

The resulting equation is

$$\mathrm{g}(x) = -3x + 3 + \frac{1}{x-1}\ .$$

Find the equation of $y = \mathrm{f}(x)$, showing your workings clearly. [4]

4 **(i)** Without using a calculator, solve the inequality,

$$\frac{x^2+7x-17}{2x-5} \geq 3. \qquad [4]$$

(ii) Solve $\dfrac{e^{-2x}+7e^{-x}-17}{2e^{-x}-5} \geq 3$, leaving your answers in exact form. [3]

5 It is given that

$$f(x)=\begin{cases} x^2+a & \text{for } 0<x\leq 2 \\ (8+a)-2x & \text{for } 2<x\leq 4 \end{cases}$$

where a is a positive constant and $f(x)=f(x+4)$ for all real values of x.

(i) Find $f(21)+f(39)$ in terms of a. [2]

(ii) Sketch the graph of $y=f(x)$ for $-3\leq x\leq 11$. [3]

6 Functions f and g are defined by

$$f: x \mapsto \ln\left[(x+1)^2+4\right] \quad \text{for } x\in\mathbb{R},\ x>-5,$$

$$g: x \mapsto x^2-4 \qquad \text{for } x\in\mathbb{R}.$$

(i) Show that fg exists. Find fg(x), stating the domain and find the exact range of fg. [5]

(ii) Give a reason why f does not have an inverse. [2]

(iii) The function f has an inverse if its domain is restricted to $x>k$.

State the least value of k for which the function f^{-1} exists. [1]

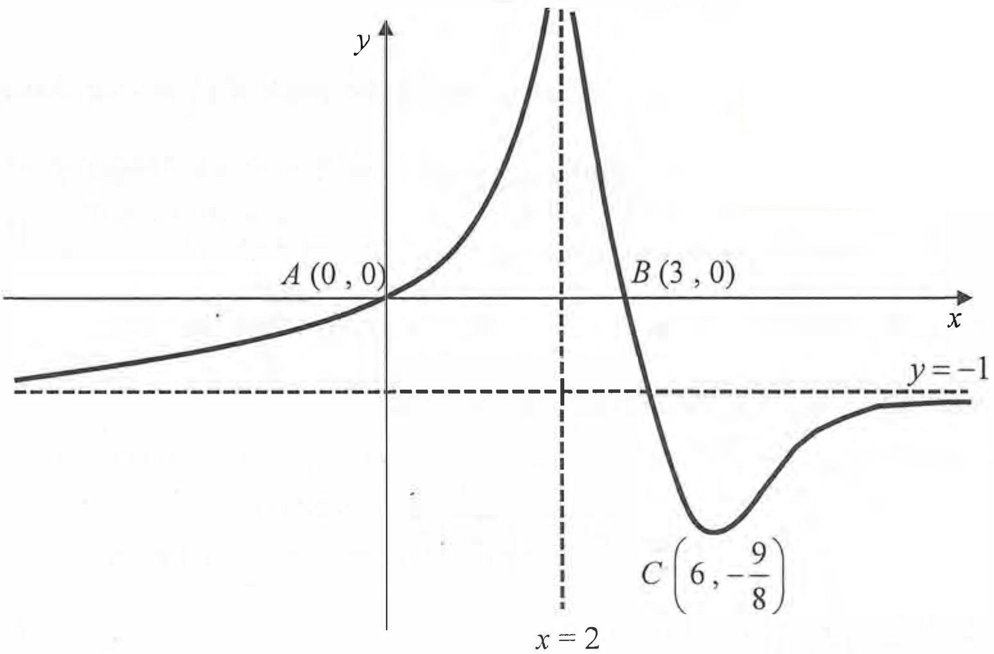
(iv) Using the value of k found in **(iii)**, find the exact range of values of x for which $f^{-1}(x)\leq 0$, showing your working clearly. [3]

7 **(i)** The graph shown in the diagram below has equation

$$h(x) = a + \frac{b-x}{(x-c)^2},$$

where a, b and c are constants.

The curve intersects the x-axis at points A (0 , 0) and B (3 , 0). It has a minimum point at $C\left(6, -\frac{9}{8}\right)$ and asymptotes $x = 2$ and $y = -1$.



Find the values of a, b and c. [4]

(ii) The graph of $y = k(x)$ has equation $y = -1 + \frac{x+3}{(x+1)^2}$.

Describe a sequence of **two** transformations that will transform the graph of $y = h(x)$ to the graph of $y = k(x)$. [3]

[Turn over

8 The curve C has equation $y = \dfrac{2x^2 - 2x - a}{x - 1}$, where $x \neq 1$ and a is a non-zero constant.

(i) Find the set of values of a for C to have no stationary points. [3]

(ii) With the set of values of a found in **(i)**, sketch C, giving the coordinates of any point(s) of intersection with the axes and the equation of any asymptote(s) in terms of a. [4]

Another curve C_1 is defined by the parametric equations

$$x = k\tan t\ ,\ \ y = m\sec t,\ \ 0 \leq t \leq 2\pi,\ t \neq \frac{\pi}{2}, \frac{3\pi}{2},$$

where m and k are positive constants.

(iii) Find the Cartesian equation of C_1. Hence, by adding the graph of C_1 in your sketch for part **(ii)**, determine the least value of $\dfrac{m}{k}$ such that C_1 intersects C exactly twice.

[6]

9 **(i)** By expressing $\dfrac{2r+3}{r(r+1)}$ in partial fractions, show that

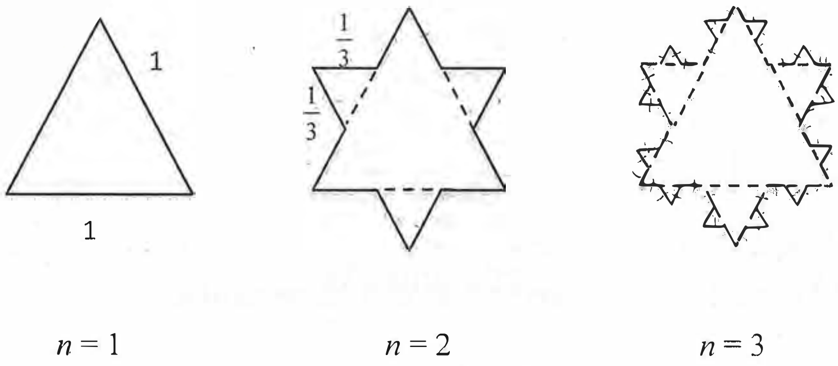
$$\frac{2r+3}{r(r+1)}\left(\frac{1}{3}\right)^r = \frac{1}{r3^{r-1}} - \frac{1}{(r+1)3^r}.$$

Hence find $\displaystyle\sum_{r=2}^{N} \frac{2r+3}{r(r+1)}\left(\frac{1}{3}\right)^r$ in terms of N. [4]

(ii) Give a reason why the series in part **(i)** is convergent and state the value of the sum to infinity. [2]

(iii) Using your answer in part **(i)**, show that $\displaystyle\sum_{r=4}^{N+2} \frac{2r+1}{r(r-1)}\left(\frac{1}{3}\right)^{r-2}$ can be expressed as $k + \mathrm{f}(N)$ where k is a constant and $\mathrm{f}(N)$ is an expression in terms of N to be determined. [5]

10 A fractal, also known as the Koch island, was first described by Helge von Koch in 1904. It is built by starting with an equilateral triangle, removing the inner third of each side, building another equilateral triangle at the location where the side was removed, and then repeating the process indefinitely.



At stage 1 (when $n = 1$) of building the fractal, an equilateral triangle of side 1m was constructed. At stage 2 (when $n = 2$), another equilateral triangle was built onto the middle third of each side of the first triangle. (See diagram above).

Let L_n represents the number of sides at the nth stage of building the fractal. For example, L_1 is 3, L_2 is 12, etc.

(i) Let P_n be the perimeter of the fractal at stage n. By copying and completing the table for $n = 3$, deduce an expression for L_n and P_n in terms of n. The first two rows have been completed for you. [4]

n	L_n	P_n
1	3	3
2	$3\times4=12$	$3\times4\times\frac{1}{3}=4$
3		
⋮	⋮	⋮
n		

(ii) What can you conclude about the perimeter of the fractal as n tends to infinity? [1]

(iii) Show that the area of **each** triangle added in stage 2 is one-ninth of the area of the first triangle. [1]

 [Turn over

(iv) Let A_n be the **area of all triangles added** at stage n. The values of L_n and A_n for $n = 1$ and 2 are shown in the table below.

	L_n	A_n
1	3	$\frac{\sqrt{3}}{4}$
2	12	$3\left[\left(\frac{1}{9}\right)\frac{\sqrt{3}}{4}\right]$

Find L_n and A_n for $n = 3$ and 4.

Hence, for $n \geq 2$, form an expression for A_n in terms of n. [4]

(v) Find the area of the fractal at stage 10. [3]

End of Paper