AJC 2017 Promo

<table>
<tr><td>1</td><td>

$y=\dfrac{x^2+x-1}{x+2}$

$xy+2y=x^2+x-1$

$x^2+x(1-y)-2y-1=0$

which is a quadratic equation with real coefficients.
Discriminant ≥ 0 as every value of y corresponds to real value(s) of x.

$(1-y)^2-4(1)(-2y-1)\geq 0$

$y^2+6y+5\geq 0$

$(y+5)(y+1)\geq 0$

$y\leq -5$ or $y\geq -1$

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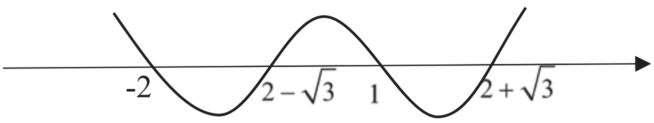
<tr><td>2</td><td>

$\dfrac{2x^2-3x-1}{x^2-4x+1}<1$

$\Rightarrow \dfrac{2x^2-3x-1-\left(x^2-4x+1\right)}{x^2-4x+1}<0 \quad \Rightarrow \dfrac{x^2+x-2}{x^2-4x+1}<0$

$\dfrac{(x-1)(x+2)}{\left(x-2\right)^2-3}<0$

$\dfrac{(x-1)(x+2)}{\left(x-2-\sqrt{3}\right)\left(x-2+\sqrt{3}\right)}<0 \quad \left[x-\left(2+\sqrt{3}\right)\right]\left[x-\left(2-\sqrt{3}\right)\right](x-1)(x+2)<0$



From sketch, $-2<x<2-\sqrt{3}$ or $1<x<2+\sqrt{3}$

Replace x in $\dfrac{2x^2-3x-1}{x^2-4x+1}<1$ by $\left(-\sqrt{x}\right)$, we get $\dfrac{2\left(-\sqrt{x}\right)^2-3\left(-\sqrt{x}\right)-1}{\left(-\sqrt{x}\right)^2-4\left(-\sqrt{x}\right)+1}<1$ which simplifies to

$\dfrac{2x+3\sqrt{x}-1}{x+4\sqrt{x}+1}<1$.

Hence, $-2<-\sqrt{x}<2-\sqrt{3}$ or $1<-\sqrt{x}<2+\sqrt{3}$.

But $1<-\sqrt{x}<2+\sqrt{3}$ has no solution for x as $-\sqrt{x}\leq 0$.

$-2<-\sqrt{x}<2-\sqrt{3} \Leftrightarrow \sqrt{3}-2<\sqrt{x}<2$

Since $\sqrt{x}\geq 0$, $\therefore 0\leq \sqrt{x}<2 \Rightarrow 0\leq x<4$

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(i) smallest possible $k=1$
(if k is anything less than 1, f will no longer be one-one)

(ii) Line to reflect in: $y=x$

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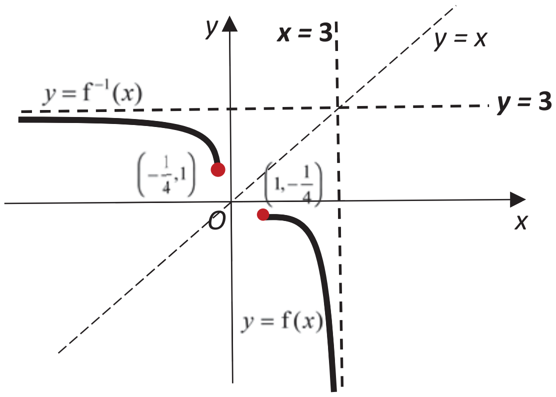


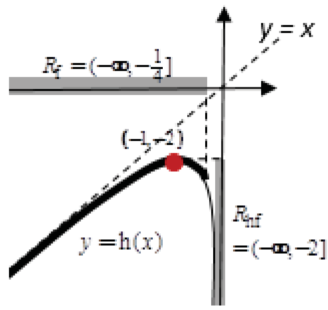
(iii) As there are **no points of intersection** between the graphs of f and f^{-1}, the number of solutions to $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ is **zero**.

(iv) $R_{\mathrm{f}}=\left(-\infty,-\dfrac{1}{4}\right]$, $\quad D_{\mathrm{h}}=(-\infty,0)$

Since $R_{\mathrm{f}}\subseteq D_{\mathrm{h}}$, hf exist.

$$\underbrace{\left(-\infty,-\tfrac{1}{4}\right]}_{R_{\mathrm{f}}}\xrightarrow{\mathrm{h}}\underbrace{(-\infty,-2]}_{R_{\mathrm{hf}}}$$

$\therefore R_{\mathrm{hf}}=(-\infty,-2]$



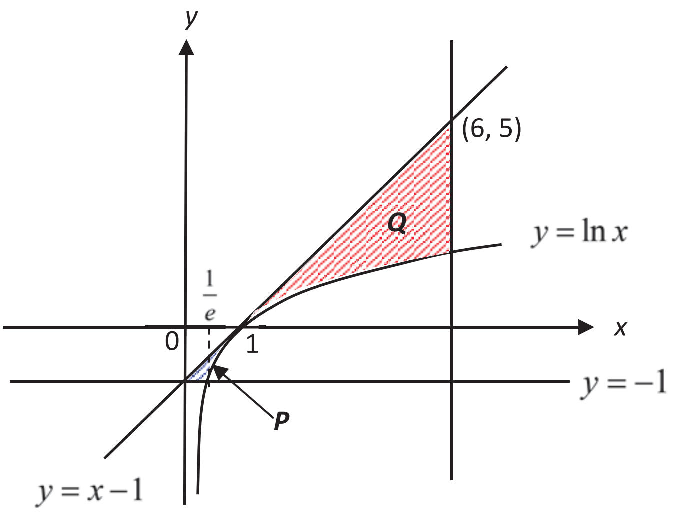
4

(a) $$\int\frac{x^2+x+1}{x^2-x+1}\,\mathrm{d}x=\int\left(1+\frac{2x}{x^2-x+1}\right)\mathrm{d}x$$
$$=\int\left(1+\frac{2x-1}{x^2-x+1}+\frac{1}{x^2-x+1}\right)\mathrm{d}x$$
$$=\int\left(1+\frac{2x-1}{x^2-x+1}+\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\right)\mathrm{d}x$$
$$=x+\ln\left|x^2-x+1\right|+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$
$$=x+\ln\left|x^2-x+1\right|+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$$

(b) $x=2\cos u\Rightarrow\dfrac{\mathrm{d}x}{\mathrm{d}u}=-2\sin u$

When $x=0,\ u=\dfrac{\pi}{2}$; $\quad$ When $x=1,\ u=\dfrac{\pi}{3}$

$$\int_0^1\sqrt{4-x^2}\,\mathrm{d}x=\int_{\frac{\pi}{2}}^{\frac{\pi}{3}}\sqrt{4-4\cos^2 u}\,(-2\sin u)\,\mathrm{d}u$$
$$=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}4\sin^2 u\,\mathrm{d}u$$

	$= 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1-\cos 2u)\,\mathrm{d}u$ $= 2\left[u - \frac{\sin 2u}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$ $= 2\left(\frac{\pi}{2} - \frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$
5	(i) For equation of tangent, $y - 0 = \frac{1}{1}(x-1)$ $y = x - 1$ (ii)  Area of region P $= \int_{-1}^{0} x\,\mathrm{d}y - \frac{1}{2}(\text{base})(\text{height})$ $= \int_{-1}^{0} \mathrm{e}^{y}\,\mathrm{d}y - \frac{1}{2}(1)(1)$ $= \left[\mathrm{e}^{y}\right]_{-1}^{0} - \frac{1}{2}$ $= \frac{1}{2} - \frac{1}{e} \quad = 0.132$ (3 s.f.) (iii) Volume generated $= \frac{1}{3}\pi(5)^2(6-1) - \pi\int_{1}^{6} (\ln x)^2\,\mathrm{d}x$ $= \frac{125}{3}\pi - \pi\left[x(\ln x)^2\right]_1^6 + \pi\int_1^6 2\ln x\,\mathrm{d}x$ $= \frac{125}{3}\pi - \pi\left[x(\ln x)^2\right]_1^6 + \pi\left[2x\ln x\right]_1^6 - \pi\int_1^6 2\,\mathrm{d}x$ $= \frac{125}{3}\pi - \pi\left[x(\ln x)^2 - 2x\ln x + 2x\right]_1^6$ $= \frac{125}{3}\pi - \pi\left[6(\ln 6)^2 - 12\ln 6 + 12 - 2\right]$ $= \frac{95}{3}\pi - 6\pi\ln 6(\ln 6 - 2)$

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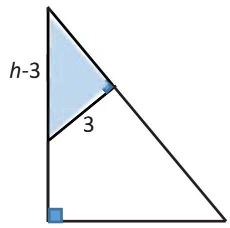
(i) By similar triangles, $\dfrac{r}{h} = \dfrac{3}{\sqrt{(h-3)^2 - 3^2}}$



$$r = \frac{3\,h}{\sqrt{h^2 - 6h}}$$

(ii) $V = \dfrac{1}{3}\pi r^2 h = \dfrac{\pi}{3}\left(\dfrac{3h}{\sqrt{h^2-6h}}\right)^2 h \quad = \dfrac{3\pi h^2}{h-6}$

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{(h-6)(6\pi h) - 3\pi h^2}{(h-6)^2}$$

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{3\pi h(h-12)}{(h-6)^2}$$

Consider $\dfrac{\mathrm{d}V}{\mathrm{d}h} = 0 \quad \Rightarrow \quad 3\pi h\,(h-12) = 0$

Since $V \neq 0 \quad , \quad h = 12$

When $\quad h = 12,$ volume $V = \dfrac{3\pi(144)}{12-6} = 72\pi$

To prove minimum, $\dfrac{\mathrm{d}^2V}{\mathrm{d}h^2} = \dfrac{(h-6)^2 3\pi(2h-12) - \big(2(h-6)\big)\, 3\pi h(h-12)}{(h-6)^4}$

When $\;h = 12\,,\;\; \dfrac{\mathrm{d}^2V}{\mathrm{d}h^2} = \dfrac{(12-6)^2 3\pi(24-12) - 0}{(12-6)^4} = \pi > 0$. Hence volume is minimum, $V_{\min} = 72\pi$.

<u>Alternatively</u>, using GC, $\dfrac{\mathrm{d}^2V}{\mathrm{d}h^2} = 3.14 > 0$ when $\;h = 12$. Hence volume is minimum, $V_{\min} = 72\pi$.

<u>Alternatively</u>, using first derivative

	$h = 12^-$	$h = 12$	$h = 12^+$
$\dfrac{\mathrm{d}V}{\mathrm{d}h} = \dfrac{3\pi h(h-12)}{(h-6)^2}$	$\dfrac{\mathrm{d}V}{\mathrm{d}h} = \dfrac{(+)(-)}{(+)} < 0$	0	$\dfrac{\mathrm{d}V}{\mathrm{d}h} = \dfrac{(+)(+)}{(+)} > 0$
slope	\	—	/

$V_{\min} = 72\pi$.

7

(i) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = e^t \cos t - e^t \sin t \quad ; \quad \dfrac{\mathrm{d}y}{\mathrm{d}t} = e^t \cos t + e^t \sin t$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{e^t(\cos t + \sin t)}{e^t(\cos t - \sin t)} = \frac{\cos t + \sin t}{\cos t - \sin t}$$

(ii) Normal parallel to y-axis = Tangent parallel to x-axis

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \Rightarrow \quad \cos t + \sin t = 0 \quad \Rightarrow \quad t = -\dfrac{\pi}{4}$ or $t = -0.785398$

Equation of Normal is $\;x = \dfrac{\sqrt{2}}{2} e^{-\frac{\pi}{4}}$ or $\;x = 0.3223969 \approx 0.322$

(iii) At point N, $e^t \cos t = \frac{\sqrt{2}}{2} e^{-\frac{\pi}{4}}$

By using GC, draw $y_1 = e^t \cos t - \frac{\sqrt{2}}{2} e^{-\frac{\pi}{4}} \Rightarrow t = 1.4987$

Hence $N \equiv (0.322, 4.464258813)$ $\therefore N \equiv (0.322, 4.464)$ to 3 d.p.

(iv) $P \equiv (0.322, -0.322)$

Area $\Delta OPN = \frac{1}{2}(0.322)(0.322 + 4.464) = 0.770546 = 0.771$

Note: if 5 d.p. is used in working, $N \equiv (0.322, 4.464)$ to 3 d.p.

Area ~ 0.772 to 3 d.p.

if more d.p. used in working, same answers as using 5 d.p.

8

(i) Given: $\mathbf{r} \bullet \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} = 24$, $\overrightarrow{OP} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$ $\overrightarrow{OQ} = \begin{pmatrix} 8 \\ 0 \\ 8 \end{pmatrix}$ $\overrightarrow{OR} = \begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix}$

$$\overrightarrow{OP} \bullet \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} = 24 = RHS$$

$$\overrightarrow{OQ} \bullet \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 8 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} = -8 + 32 = 24 = RHS$$

$$\overrightarrow{OR} \bullet \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} = -16 + 40 = 24 = RHS$$

Thus plane passes through P, Q and R.

(ii) Let θ be the angle between the plane PQR and the horizontal.

$$\cos\theta = \frac{\left| \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|}{\sqrt{21}} = \frac{4}{\sqrt{21}} \Rightarrow \theta = 29.2^\circ$$

Plane PQR: $\mathbf{r} \bullet \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} = 24$

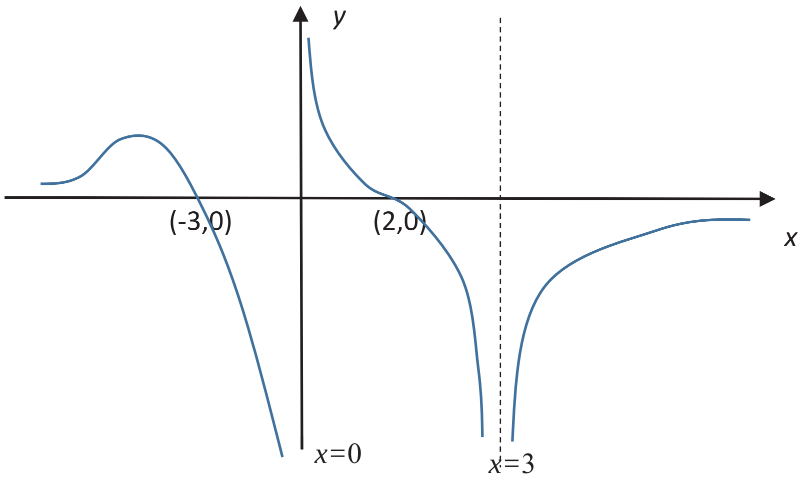
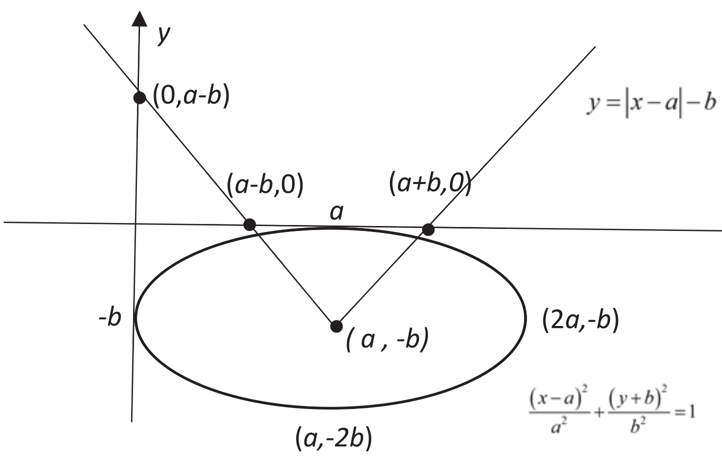
Let the foot of perpendicular from point A to plane PQR be F.

$l_{AF} : \underset{\sim}{r} = \begin{pmatrix} 8 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix}$ $\lambda \in \mathbb{R}$. At F, $\begin{pmatrix} 8-\lambda \\ -2\lambda \\ 4\lambda \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} = 24$

$\Rightarrow -8 + \lambda + 4\lambda + 16\lambda = 24 \Rightarrow 21\lambda = 32 \Rightarrow \lambda = \frac{32}{21}$ $\overrightarrow{OF} = \frac{1}{21}\begin{pmatrix} 136 \\ -64 \\ 128 \end{pmatrix}$

(iii) By Ratio Theorem, $\overrightarrow{OF} = \frac{\overrightarrow{OA} + \overrightarrow{OA}'}{2} \Rightarrow \overrightarrow{OA}' = 2\overrightarrow{OF} - \overrightarrow{OA}$

	Thus, $\overrightarrow{OA}' = 2\overrightarrow{OF} - \overrightarrow{OA} = \frac{2}{21}\begin{pmatrix}136\\-64\\128\end{pmatrix} - \begin{pmatrix}8\\0\\0\end{pmatrix} = \frac{8}{21}\begin{pmatrix}13\\-16\\32\end{pmatrix}$ (iv) Distance between point N and the plane PQR = 12 $\Rightarrow \left\| \begin{pmatrix}\alpha\\9\\0\end{pmatrix} \bullet \frac{1}{\sqrt{21}}\begin{pmatrix}-1\\-2\\4\end{pmatrix} - \frac{24}{\sqrt{21}} \right\| = 12$ $\Rightarrow \|-\alpha - 18 - 24\| = 12\sqrt{21} \quad \Rightarrow -\alpha - 42 = \pm 12\sqrt{21}$ $\Rightarrow \quad -\alpha = 42 \pm 5\sqrt{21} \Rightarrow \quad \alpha = 12.990 = 13.0$ since $\alpha > 0$
9	(i) $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = \mathbf{a} + \lambda\overrightarrow{AB} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$ $\overrightarrow{OX} = \overrightarrow{OC} + \overrightarrow{CX} = \frac{2}{5}\mathbf{b} + \mu\overrightarrow{CD} = \frac{3}{2}\mathbf{a} + \mu(\frac{2}{5}\mathbf{b} - \frac{3}{2}\mathbf{a})$ $\therefore \quad \overrightarrow{OX} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) = \frac{3}{2}\mathbf{a} + \mu(\frac{2}{5}\mathbf{b} - \frac{3}{2}\mathbf{a})$ $(1-\lambda)\mathbf{a} + \lambda\mathbf{b} = \left(\frac{3}{2} - \frac{3}{2}\mu\right)\mathbf{a} + \frac{2}{5}\mu\mathbf{b}$ Since **a** and **b** are non-zero and non-parallel vectors, $1-\lambda = \frac{3}{2} - \frac{3}{2}\mu$ **&** $\lambda = \frac{2}{5}\mu \quad \Rightarrow \quad \lambda = \frac{2}{11}, \quad \mu = \frac{5}{11}$ $\overrightarrow{OX} = \mathbf{a} + \left(\frac{2}{11}\right)(\mathbf{b} - \mathbf{a}) = \frac{9}{11}\mathbf{a} + \frac{2}{11}\mathbf{b}$ (shown) (ii) $\overrightarrow{AD} = \frac{2}{5}\mathbf{b} - \mathbf{a}$; $\quad \overrightarrow{AX} = \frac{2}{11}(\mathbf{b} - \mathbf{a})$ Area of triangle ADX = $\frac{1}{2}\left\|\overrightarrow{AD} \times \overrightarrow{AX}\right\| = \frac{1}{2}\left\|\left(\frac{2}{5}\mathbf{b} - \mathbf{a}\right) \times \frac{2}{11}(\mathbf{b} - \mathbf{a})\right\|$ $= \frac{1}{11}\left\|\left(\frac{2}{5}\mathbf{b} - \mathbf{a}\right) \times (\mathbf{b} - \mathbf{a})\right\|$ $= \frac{1}{11}\left\|-\frac{2}{5}(\mathbf{b} \times \mathbf{a}) - (\mathbf{a} \times \mathbf{b})\right\|$ (since $\mathbf{a} \times \mathbf{a} = \underset{\sim}{0}$ and $\mathbf{b} \times \mathbf{b} = \underset{\sim}{0}$) $= \frac{1}{11}\left\|\frac{2}{5}(\mathbf{a} \times \mathbf{b}) - (\mathbf{a} \times \mathbf{b})\right\|$ $= \frac{3}{55}\|\mathbf{a} \times \mathbf{b}\|$

10	 (a) (b) $y = ax^3 + bx^2 + cx + \frac{32}{3}$ At $(1,1)$, $a+b+c+\frac{32}{3}=1 \ \therefore a+b+c=-\frac{29}{3}$ ---------------------(1) $\frac{dy}{dx} = 3ax^2 + 2bx + c$ $\frac{dy}{dx} = 0$ at $x=2 \ \therefore 12a+4b+c=0$ --------------------------(2) Undergoing transformation, the resultant equation will be $y = -8ax^3 - 4bx^2 - 2cx - \frac{32}{3}$ Sub $(2,-16)$, $-64a-16b-4c-\frac{32}{3}=-16$ $\therefore 64a+16b+4c=\frac{16}{3}$ ---------------------(3) Solving, $a=\frac{1}{3}$, $b=2$, $c=-12$ <u>Alternatively</u>, the point $(2,-16)$ before transformation was at $(4,16)$ $64a+16b+4c+\frac{32}{3}=16 \Rightarrow 64a+16b+4c=\frac{16}{3}$
11	 (ii) A <u>translation</u> of magnitude *a* units in the positive direction of the *x*-axis.

	A <u>translation</u> of magnitude b units in the negative direction of the y-axis. (iii) Observe $y=-b+\frac{b}{a}\sqrt{a^2-(x-a)^2} \Rightarrow \frac{(x-a)^2}{a^2}+\frac{(y+b)^2}{b^2}=1$ Thus $y=-b+\frac{b}{a}\sqrt{a^2-(x-a)^2}$ is the top half of the ellipse, and is in the 4th quadrant. Hence $y=-b+\frac{b}{a}\sqrt{a^2-(x-a)^2} \le 0$ for $0 \le x \le 2a$. For $\left(\lvert x-a\rvert-b\right)\left(-b+\frac{b}{a}\sqrt{a^2-(x-a)^2}\right)<0$, $y=\lvert x-a\rvert-b>0$ for $0 \le x<1$ or $5<x \le 2a$. With the help of the diagram, $a-b=1$ and $a+b=5$. solving : $a=3,\ b=2$