O-LEVEL A-MATHS 2016 – PAPER 2

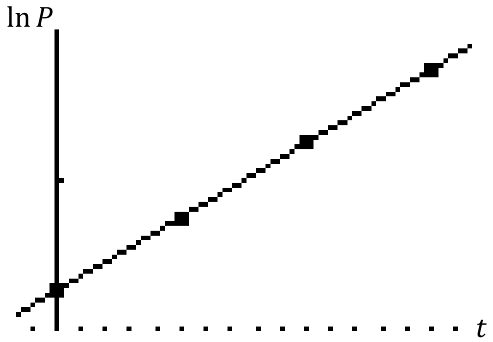
Question 1

[Ans: (i) plot (ii) $P_o \approx 2.00$, $k \approx 0.0402$ (iii) $P \approx 4.47$]

(i) $P = P_0 e^{kt}$
$\ln P = \ln\left(P_0 e^{kt}\right)$
$\ln P = \ln P_0 + \ln e^{kt}$
$\ln P = \ln P_0 + kt \ln e$
$\ln P = \ln P_0 + kt$

t	0	5	10	15
$\ln P$	0.693	0.892	1.10	1.29

Plotting on a graph paper:



(ii) $\ln P_0 =$ vertical-intercept
$\ln P_0 = 0.693$
$P_0 = e^{0.693} = 2.00$

$k =$ gradient of line $= 0.0402$

(iii) $\ln P = 0.693 + 0.0402t$
On 1st Jan 2015,
$t = 20$

$\Rightarrow \ln P = 0.693 + 0.0402(20)$
$\ln P = 1.497$
$P = e^{1.497} = 4.47$

Question 2

[Ans: (i) $p = -2$ or $p = \dfrac{2}{5}$ (ii) 160 or $-\dfrac{32}{25}$]

(i) $(1-2x)^2(1-px)^6$

$= (1-4x+4x^2)\left[1^6 + \binom{6}{1}(1)^5(-px) + \binom{6}{2}(1)^4(-px)^2 + \cdots\right]$

$= (1-4x+4x^2)(1-6px+15p^2x^2-\cdots)$

$= (1)(15p^2x^2) + (-4x)(-6px) + (4x^2)(1) + \cdots$

$= (15p^2+24p+4)x^2 + \cdots$

$\therefore 15p^2+24p+4 = 16$

$15p^2+24p-12 = 0$

$5p^2+8p-4 = 0$

$(p+2)(5p-2) = 0$

$p = -2$ or $p = \dfrac{2}{5}$

(ii) Term in x^3 of $(1-px)^6$

$= \binom{6}{3}(1)^3(-px)^3$

$= -20p^3x^3$

For $p = -2$,

coefficient of x^3

$= -20(-2)^3 = 160$

For $p = \dfrac{2}{5}$,

coefficient of x^3

$= -20\left(\dfrac{2}{5}\right)^3 = -\dfrac{32}{25}$

Question 3

[Ans: (i) show (ii) $x = 7.2°, 90°, 172.8°$ or $270°$]

(i) $\cos 3x$
$= \cos(2x + x)$
$= \cos 2x \cos x - \sin 2x \sin x$
$= (1 - 2\sin^2 x)\cos x - (2\sin x \cos x)\sin x$
$= \cos x\,(1 - 2\sin^2 x - 2\sin^2 x)$
$= \cos x\,(1 - 4\sin^2 x)$ (shown)

(ii) $2\cos 3x = 15\sin x \cos x$
$2\cos x\,(1 - 4\sin^2 x) = 15\sin x \cos x$
$2\cos x\,(1 - 4\sin^2 x) - 15\sin x \cos x = 0$
$\cos x\,[2(1 - 4\sin^2 x) - 15\sin x] = 0$
$\cos x\,(2 - 8\sin^2 x - 15\sin x) = 0$
$\cos x\,(2 + \sin x)(1 - 8\sin x) = 0$

$\cos x = 0$ or $\sin x = -2$ (NA) or $\sin x = \frac{1}{8}$

$x = 90°, 270°$

Basic ∠
$= \sin^{-1}\frac{1}{8} = 7.1507°$

$x = 7.1507°, 180° - 7.1507°$
$= 7.2°, 172.8°$

Question 4

[Ans: (i) 22 (ii) $25x^2 - 22x + 5 = 0$]

(i) Given $x^2 + 2x + 5 = 0$
$\alpha + \beta = -\frac{2}{1} = -2, \qquad \alpha\beta = \frac{5}{1} = 5$

$(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$
$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\,\alpha\beta(\alpha + \beta)$
$= (-2)^3 - 3(5)(-2)$
$= 22$

(ii) Sum of new roots
$= \frac{\alpha}{\beta^2} + \frac{\beta}{\alpha^2}$
$= \frac{\alpha^3 + \beta^3}{(\alpha\beta)^2}$
$= \frac{22}{5^2} = \frac{22}{25}$

Product of new roots
$= \left(\frac{\alpha}{\beta^2}\right)\left(\frac{\beta}{\alpha^2}\right)$
$= \frac{1}{\alpha\beta} = \frac{1}{5}$

New equation:
$x^2 - \frac{22}{25}x + \frac{1}{5} = 0$
$25x^2 - 22x + 5 = 0$

Queston 5

[Ans: (i) prove (ii) prove]

(i) $\angle PAB = \angle ACB$ (alt. segment th.)
$\angle PBA = \angle ACB$ (alt. segment th.)

$\angle ACB = \angle DCB = \angle DBC$ (base ∠s of isos. Δ)

$\angle ADB$
$= \angle DCB + \angle DBC$ (ext. ∠ of Δ)
$= \angle ACB + \angle ACB = 2\angle ACB$

$\angle APB$
$= 180° - \angle PAB - \angle PBA$ (∠ sum of Δ)
$= 180° - \angle ACB - \angle ACB$
$= 180° - 2\angle ACB$
$= 180° - \angle ADB$

$\therefore \angle APB = 180° - \angle ADB$
$\angle APB + \angle ADB = 180°$ (proven)

(ii) $\angle PAB = \angle PDB$ (∠s in same segment)

$\angle PAB = \angle ACB$ (alt. segment th.)

$\therefore \angle PDB = \angle ACB$
⇒ PD and BC are parallel (alt. ∠s)

Question 6

[Ans: (i) $(8x-17)(2x-5)^2$ (ii) $x<\frac{17}{8}$ (iii) 0.05 units per second (iv) show]

(i) $y=(x-2)(2x-5)^3$

$\frac{dy}{dx}=(x-2)[3(2x-5)^2(2)]+(1)(2x-5)^3$
$=6(x-2)(2x-5)^2+(2x-5)^3$
$=[6(x-2)+(2x-5)](2x-5)^2$
$=(6x-12+2x-5)(2x-5)^2$
$=(8x-17)(2x-5)^2$

(ii) For y being a decreasing function,

$\frac{dy}{dx}<0$
$(8x-17)(2x-5)^5<0$
$8x-17<0$
$x<\frac{17}{8}$

(iii) When $x=3$,

$\frac{dy}{dt}=0.35$ (given)

$\frac{dy}{dx}=[8(3)-17][2(3)-5]^2$
$=7$

$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$
$0.35=7\frac{dx}{dt}$
$\frac{dx}{dt}=0.05$

(iv) $z=y^2$

$\frac{dz}{dy}=2y$

When $x=3$,

$y=(3-2)[2(3)-5]^3=1$
$\frac{dz}{dy}=2(1)=2$
$\therefore\frac{dz}{dt}=\frac{dz}{dy}\times\frac{dy}{dt}$
$\frac{dz}{dt}=2\frac{dy}{dt}$ (shown)

Question 7

[Ans: (i) $u^2 - 8u + 12 = 0$ (ii) $x = 1$ or 2.6 (iii) explain]

(i) $2^{2x-1} = 2^{x+2} - 6$

$2^{2x}2^{-1} = 2^x 2^2 - 6$

$\frac{1}{2}(2^x)^2 = 4(2^x) - 6$

$\frac{1}{2}u^2 = 4u - 6$

$u^2 = 8u - 12$

$u^2 - 8u + 12 = 0$

(ii) $(u-2)(u-6) = 0$

$u = 2$	or	$u = 6$
$2^x = 2$		$2^x = 6$
$x = 1$		$\ln 2^x = \ln 6$
		$x \ln 2 = \ln 6$
		$x = \frac{\ln 6}{\ln 2} = 2.6$

(iii) $2^{2x-1} = 2^{x+2} - k$

$\frac{1}{2}u^2 = 4u - k$

$u^2 = 8u - 2k$

$u^2 - 8u + 2k = 0$

Discriminant

$= (-8)^2 - 4(1)(2k)$

$= 64 - 8k$

$= 8(8 - k)$

If $k > 8$,

$8 - k < 0$

$8(8 - k) < 0$

$\Rightarrow$ Discriminant < 0

$\therefore u^2 - 8u + 2k = 0$ will have no solution.

$\therefore 2^{2x-1} = 2^{x+2} - k$ will also have no solution.

Question 8

[Ans: (i) $(x-3)(x^2+4)$ (ii) explain; 3 (iii) $k=-1$]

(i) Given $f(x) = x^3 - 3x^2 + 4x - 12$

By trial and error,

$f(3) = 3^3 - 3(3)^2 + 4(3) - 12 = 0$

$\therefore x - 3$ is a factor of $f(x)$.

$$\begin{array}{r}
x^2 + 0x + 4 \\
x - 3 \,|\, x^3 - 3x^2 + 4x - 12 \\
\underline{-(x^3 - 3x^2)} \\
0 + 4x - 12 \\
\underline{-(4x - 12)} \\
0
\end{array}$$

$\therefore f(x) = (x-3)(x^2+4)$

(ii) For $f(x) = 0$,

$x - 3 = 0$ or $x^2 + 4 = 0$ (NA)

$\therefore f(x) = 0$ has only one real root.

The real root is 3

(iii) $y = f(x) + kx$

$y = x^3 - 3x^2 + 4x - 12 + kx$

$y = x^3 - 3x^2 + (4+k)x - 12$

$$\frac{dy}{dx} = 3x^2 - 6x + (4+k)$$

$$\frac{d^2y}{dx^2} = 6x - 6$$

Let $\frac{d^2y}{dx^2} = 0$

$6x - 6 = 0$

$6x = 6 \Rightarrow x = 1$

When $x = 1$,

$\frac{dy}{dx} = 0$

$3(1)^2 - 6(1) + (4+k) = 0$

$1 + k = 0$

$k = -1$

Question 9

[Ans: (i) 1 (ii) -2 (iii) $\frac{628}{81}$ units^2]

(i) $\frac{dy}{dx} = 3x^2 + 4x - 3$

At point A,

$$\frac{dy}{dx} = 3\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right) - 3 = 1$$

∴ gradient of the curve at A is 1.

(ii) Let $\frac{dy}{dx} = 1$

$3x^2 + 4x - 3 = 1$

$3x^2 + 4x - 4 = 0$

$(3x - 2)(x + 2) = 0$

$x = \frac{2}{3}$ or $x = -2$

∴ x-coordinate of $B = -2$

(iii) Total area

$$= \int_{-2}^{0} x^3 + 2x^2 - 3x\,dx + \left|\int_{0}^{\frac{2}{3}} x^3 + 2x^2 - 3x\,dx\right|$$

$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{-2}^{0} + \left|\left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{0}^{\frac{2}{3}}\right|$$

$$= \left\{(0) - \left[\frac{(-2)^4}{4} + \frac{2(-2)^3}{3} - \frac{3(-2)^2}{2}\right]\right\} + \left|\left[\frac{\left(\frac{2}{3}\right)^4}{4} + \frac{2\left(\frac{2}{3}\right)^3}{3} - \frac{3\left(\frac{2}{3}\right)^2}{2}\right] - 0\right|$$

$$= \left[-\left(-\frac{22}{3}\right)\right] + \left|-\frac{34}{81}\right|$$

$$= \frac{22}{3} + \frac{34}{81}$$

$$= \frac{628}{81} \text{ units}^2$$

Question 10

[Ans: (i) 50 (ii) 100s (iii) 1.75km (iv) $750e^{25t}$]

(i) At A, $t = 0$

$v = p$

$30e^{25(0)} + 20 = p$

$p = 30 + 20 = 50$

(ii) At B,

$v = 80$

$30e^{25t} + 20 = 80$

$30e^{25t} = 60$

$e^{25t} = 2$

$\ln e^{25t} = \ln 2$

$25t = \ln 2$

$t = \frac{\ln 2}{25}$

$= 0.027726\text{h}$

$= 0.027726 \times 60 \times 60 = 100s$

(iii) $v = 30e^{25t} + 20$

$s = \int 30e^{25t} + 20\, dt$

$= 30\left(\frac{e^{25t}}{25}\right) + 20t + c$

$= \frac{6}{5}e^{25t} + 20t + c$

When $t = 0$,

$s = \frac{6}{5}e^{25(0)} + 20(0) + c = \frac{6}{5} + c$

When $t = \frac{\ln 2}{25}$,

$s = \frac{6}{5}e^{25\left(\frac{\ln 2}{25}\right)} + 20\left(\frac{\ln 2}{25}\right) + c = 2.9545 + c$

∴ distance between A and B

$= (2.9545 + c) - \left(\frac{6}{5} + c\right)$

$= 1.75$

(iv) $v = 30e^{25t} + 20$

$a = \frac{dv}{dt}$

$= 30e^{25t}(25) = 750e^{25t}$

Question 11

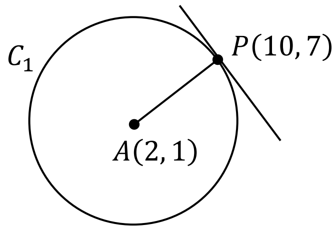
[Ans: (i) $A(2,1)$, radius $= 10$ (ii) show (iii) $y = -\frac{4}{3}x + \frac{61}{3}$ (iv) $x^2 + y^2 - 12x - 8y = -27$

(v) $y = -\frac{4}{3}x + \frac{61}{3}$]

(i) $x^2 + y^2 - 4x - 2y = 95$
$(x-2)^2 - 2^2 + (y-1)^2 - 1^2 = 95$
$(x-2)^2 + (y-1)^2 = 100$
$(x-2)^2 + (y-1)^2 = 10^2$
$\therefore A(2,1)$ and radius of C_1 is 10.

(ii) At P,
$10^2 + 7^2 - 4(10) - 2(7) = 95$
$\therefore P$ lies on C_1.

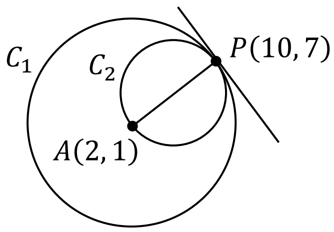
(iii)



Gradient of AP
$= \frac{7-1}{10-2} = \frac{3}{4}$

Equation of tangent to C_1 at P:
$y - 7 = -\frac{1}{3/4}(x - 10)$
$y - 7 = -\frac{4}{3}x + \frac{40}{3} \Rightarrow y = -\frac{4}{3}x + \frac{61}{3}$

(iv)



Centre of C_2
$= \left(\frac{2+10}{2}, \frac{1+7}{2}\right) = (6,4)$

Radius of C_2
$= \frac{10}{2} = 5$

Equation of C_2:
$(x-6)^2 + (y-4)^2 = 5^2$
$x^2 - 12x + 36 + y^2 - 8y + 16 = 25$
$x^2 + y^2 - 12x - 8y = -27$

(v) Equation of tangent to C_2 at P:
$y = -\frac{4}{3}x + \frac{61}{3}$