O-LEVEL A-MATHS 2015 – PAPER 2

Question 1

(i) e^x and e^{-2x} are positive for all real values of x.
$\therefore f'(x) = 2e^x + e^{-2x}$ is positive for all real values of x.
$\Rightarrow f(x)$ has no stationary points.

(ii) $f(x)$

$$= \int 2e^x + e^{-2x}\,dx$$

$$= 2e^x + \frac{e^{-2x}}{-2} + c = 2e^x - \frac{1}{2}e^{-2x} + c$$

At $(0, 2)$,

$$f(0) = 2$$

$$2e^0 - \frac{1}{2}e^0 + c = 2$$

$$c = \frac{1}{2}$$

$$\therefore f(x) = 2e^x - \frac{1}{2}e^{-2x} + \frac{1}{2}$$

Question 2

(i) $\frac{d}{dx}(\ln(\cos x))$

$$= \frac{1}{\cos x}(-\sin x)$$

$= -\tan x$ (shown)

(ii) $\frac{d}{dx}(x\tan x)$

$$= x(\sec^2 x) + (1)\tan x$$

$$= x\sec^2 x + \tan x$$

(iii) From (ii),

$$\int x\sec^2 x + \tan x\,dx = x\tan x + c$$

$$\int x\sec^2 x\,dx + \int \tan x\,dx = x\tan x + c$$

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx + c$$

$$= x\tan x + \int -\tan x\,dx + c$$

$$= x\tan x + \ln(\cos x) + c$$

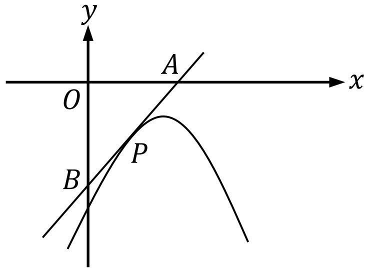
$$\int_0^{\frac{\pi}{4}} \sec^2 x\,dx = [x\tan x + \ln(\cos x)]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{4}\tan\frac{\pi}{4} + \ln\left(\cos\frac{\pi}{4}\right)\right] - [0 + \ln(\cos 0)]$$

$$= \left[\frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}\right] - (\ln 1)$$

$$= \frac{\pi}{4} + \ln 2^{-\frac{1}{2}} = \frac{\pi}{4} - \frac{1}{2}\ln 2 \quad \text{(shown)}$$

Question 3



(i) Given $y = -x^2 + 4x - 6$

$\frac{dy}{dx} = -2x + 4$

At P,

$y = -(1)^2 + 4 - 6 = -3$

$\frac{dy}{dx} = -2(1) + 4 = 2$

Equation of tangent:

$y - (-3) = 2(x - 1)$

$y = 2x - 5$

At A,

$y = 0$

$2x - 5 = 0 \Rightarrow x = \frac{5}{2}$

At B, $y = -5$

∴ Area of triangle AOB

$= \frac{1}{2}\left(\frac{5}{2}\right)(5) = \frac{25}{4}\ \text{units}^2$

(ii) At Q,

$\frac{dy}{dx} = -\frac{1}{2}$

$-2x + 4 = -\frac{1}{2} \Rightarrow x = \frac{9}{4}$

$y = -\left(\frac{9}{4}\right)^2 + 4\left(\frac{9}{4}\right) - 6 = -\frac{33}{16}$

$\therefore Q\left(\frac{9}{4}, -\frac{33}{16}\right)$

Question 4

(a) (i) $(1+x)^9$

$= 1 + 9x + \binom{9}{2}x^2 + \binom{9}{3}x^3 + \cdots$

$= 1 + 9x + 36x^2 + 84x^3 + \cdots$

(ii) $(1+z-z^2)^9$

$= [1+(z-z^2)]^9$

$= 1 + 9(z-z^2) + 36(z-z^2)^2 + 84(z-z^2)^3 + \cdots$

$= 1 + 9z - 9z^2 + 36(z^2 - 2z^3 + \cdots) + 84(z^3 + \cdots) + \cdots$

$= 1 + 9z - 9z^2 + 36z^2 - 72z^3 + 84z^3 + \cdots$

$= 1 + 9z + 27z^2 + 12z^3 + \cdots$

∴ Coefficient of $z^3 = 12$

(b) (i) T_{r+1} term

$= \binom{10}{r}(2x)^{10-r}\left(\frac{1}{3x^3}\right)^r$

$= \binom{10}{r}(2)^{10-r}\left(\frac{1}{3}\right)^r x^{10-r}x^{-3r} = \binom{10}{r}(2)^{10-r}\left(\frac{1}{3}\right)^r x^{10-4r}$

(ii) Power of $x = 10 - 4r$

(iii) Let $10 - 4r = 2 \Rightarrow r = 2$

∴ Coefficient of x^2

$= \binom{10}{2}(2)^{10-2}\left(\frac{1}{3}\right)^2 = 1280$

Question 5

(i) $\frac{11\sqrt{3}}{2\sqrt{3}+1}$

$= \frac{11\sqrt{3}}{2\sqrt{3}+1}\left(\frac{2\sqrt{3}-1}{2\sqrt{3}-1}\right)$

$= \frac{22(3)-11\sqrt{3}}{\left(2\sqrt{3}\right)^2-1^2}$

$= \frac{66-11\sqrt{3}}{12-1} = 6-\sqrt{3}$

(ii) $AC^2 = BC^2 + AB^2$

$BC^2 = AC^2 - AB^2$

$= \left(\frac{11\sqrt{3}}{2\sqrt{3}+1}\right)^2 - \left(\sqrt{3}+1\right)^2 = \left(6-\sqrt{3}\right)^2 - \left(\sqrt{3}+1\right)^2$

$= \left(36-12\sqrt{3}+3\right) - \left(3+2\sqrt{3}+1\right)$

$= 35-14\sqrt{3}$

(iii) Let x be the length of each side of the square base.

$x^2 + x^2 = BC^2$

$x^2 = \frac{BC^2}{2} = \frac{35-14\sqrt{3}}{2}$

Volume of cuboid

$= (x)(x)\left(\sqrt{3}+1\right) = x^2\left(\sqrt{3}+1\right)$

$= \left(\frac{35-14\sqrt{3}}{2}\right)\left(\sqrt{3}+1\right)$

$= \frac{1}{2}\left(35\sqrt{3}+35-14(3)-14\sqrt{3}\right)$

$= \frac{1}{2}\left(21\sqrt{3}-7\right) = \frac{7}{2}\left(3\sqrt{3}-1\right)$ units^3

Question 6

(i) $y = \frac{2x^2}{x-1}$

$$\frac{dy}{dx} = \frac{(x-1)(4x) - (1)(2x^2)}{(x-1)^2}$$
$$= \frac{4x^2 - 4x - 2x^2}{(x-1)^2}$$
$$= \frac{2x^2 - 4x}{(x-1)^2} = \frac{2x(x-2)}{(x-1)^2}$$

At stationary points of the curve,

$$\frac{dy}{dx} = 0$$
$$\frac{2x(x-2)}{(x-1)^2} = 0$$
$$x(x-2) = 0$$

$x = 0$ or $x = 2$

When $x = 0$,

$$y = 0$$

When $x = 2$,

$$y = \frac{2(2)^2}{2-1} = 8$$

∴ Coordinates of stationary points of the curve are $(0, 0)$ and $(2, 8)$.

(ii) $\frac{dy}{dx} = \frac{2x^2 - 4x}{(x-1)^2}$

$$\frac{d^2y}{dx^2} = \frac{(x-1)^2(4x-4) - 2(x-1)(2x^2 - 4x)}{(x-1)^4}$$
$$= \frac{4(x-1)^3 - 4(x-1)(x^2 - 2x)}{(x-1)^4}$$
$$= \frac{4(x-1)[(x-1)^2 - (x^2 - 2x)]}{(x-1)^4}$$
$$= \frac{4(x^2 - 2x + 1 - x^2 + 2x)}{(x-1)^3}$$
$$= \frac{4}{(x-1)^3}$$

When $x = 0$,

$$\frac{d^2y}{dx^2} = \frac{4}{(-1)^3} = -4 < 0$$

∴ $(0, 0)$ is a maximum point.

When $x = 2$,

$$\frac{d^2y}{dx^2} = \frac{4}{(2-1)^3} = 4 > 0$$

∴ $(2, 8)$ is a minimum point.

Question 7

(i) The x- and y- coordinates of the centre of C are of the same value.

Let the centre of the circle be (a, a), where $a < 8$ since it is below and to the left of $(9, 8)$.

Radius of $C = a$

Distance from centre of C to $(9, 8)$ = Radius of C

$\sqrt{(a-9)^2 + (a-8)^2} = a$

$a^2 - 18a + 81 + a^2 - 16a + 64 = a^2$

$a^2 - 34a + 145 = 0$

$(a-5)(a-29) = 0$

$a = 5$ or $a = 29$ (NA $\because a < 8$)

Equation of C:

$$(x-5)^2 + (y-5)^2 = 25$$

(ii) Gradient of line passing through centre of C and $(9, 8)$

$= \frac{9-5}{8-5} = \frac{4}{3}$

Equation of T:

$$y - 8 = -\frac{3}{4}(x-9)$$

$$y = -\frac{3}{4}x + \frac{59}{4}$$

Question 8

(i) Let $f(x) = 2x^3 - 3x^2 - 5$

Remainder required

$= f\left(-\frac{1}{2}\right)$

$= 2\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 - 5 = -6$

(ii) Let $g(x) = 2x^3 - 3x^2 + 1$

By trial and error,

$g(1) = 2 - 3 + 1 = 0$

$\therefore$ $(x-1)$ is a factor.

$$\begin{array}{r} 2x^2 - x - 1 \\ x-1 \,|\, 2x^3 - 3x^2 + 0x + 1 \\ \underline{-(2x^3 - 2x^2)} \\ -x^2 + 0x \\ \underline{-(-x^2 + x)} \\ -x + 1 \\ \underline{-(-x + 1)} \\ 0 \end{array}$$

$\therefore g(x) = (x-1)(2x^2 - x - 1)$

$= (x-1)(x-1)(2x+1)$

$= (x-1)^2(2x+1)$

(iii) $\frac{4-5x-8x^2}{2x^3-3x^2+1}$

$= \frac{4-5x-8x^2}{(x-1)^2(2x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2x+1}$

$4 - 5x - 8x^2 = A(x-1)(2x+1) + B(2x+1) + C(x-1)^2$

Let $x = 1$,

$4 - 5 - 8 = 0 + B(2+1) + 0 \Rightarrow B = -3$

Let $x = -\frac{1}{2}$,

$4 - 5\left(-\frac{1}{2}\right) - 8\left(-\frac{1}{2}\right)^2 = 0 + 0 + C\left(-\frac{1}{2} - 1\right)^2 \Rightarrow C = 2$

Let $x = 0$,

$4 = A(-1)(1) - 3(1) + 2(-1)^2 \Rightarrow A = -5$

$\therefore \frac{4-5x-8x^2}{2x^3-3x^2+1} = -\frac{5}{x-1} - \frac{3}{(x-1)^2} + \frac{2}{2x+1}$

Question 9

(i) From triangle ABD,

$$\sin\theta = \frac{BD}{80} \Rightarrow BD = 80\sin\theta$$
$$\cos\theta = \frac{AD}{80} \Rightarrow AD = 80\cos\theta$$

$\therefore L = AB + BC + AC + BD$
$= 80 + 80 + 2(80\cos\theta) + 80\sin\theta$
$= 160 + 80\sin\theta + 160\cos\theta$ (shown)

$p = 160$, $q = 80$, $r = 160$

(ii) Let $80\sin\theta + 160\cos\theta = R\sin(\theta+\alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$
$\therefore R\cos\alpha = 80$ (1)
$R\sin\alpha = 160$ (2)

$\frac{(2)}{(1)}$ $\tan\alpha = \frac{160}{80} \Rightarrow \alpha = \tan^{-1}2 = 1.1071$

$(1)^2 + (2)^2$ $R^2 = 80^2 + 160^2 = 32000$
$R = \sqrt{32000} = 80\sqrt{5}$

$\therefore L = 160 + 80\sqrt{5}\sin(\theta + 1.11)$

(iii) $L = 310$
$160 + 80\sqrt{5}\sin(\theta + \tan^{-1}2) = 310$
$$\sin(\theta + \tan^{-1}2) = \frac{15}{8\sqrt{5}}$$
$$\theta + \tan^{-1}2 = \pi - \sin^{-1}\frac{15}{8\sqrt{5}}$$
$$\theta = \left(\pi - \sin^{-1}\frac{15}{8\sqrt{5}}\right) - \tan^{-1}2$$
$= 1.04$

Question 10

(i) $\alpha+\beta=-\left(\frac{-6}{2}\right)=3$

$\alpha\beta=\frac{5}{2}$

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$=(3)^2-2\left(\frac{5}{2}\right)=4$

(ii) $\alpha^3+\beta^3$

$=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$

$=(3)\left(4-\frac{5}{2}\right)=\frac{9}{2}$ (shown)

(iii) Sum of new roots

$=(\alpha^2+\beta)+(\alpha+\beta^2)$

$=(\alpha^2+\beta^2)+(\alpha+\beta)$

$=4+3=7$

Product of new roots

$=(\alpha^2+\beta)(\alpha+\beta^2)$

$=\alpha^3+\alpha^2\beta^2+\alpha\beta+\beta^3$

$=(\alpha^3+\beta^3)+(\alpha\beta)^2+\alpha\beta$

$=\frac{9}{2}+\left(\frac{5}{2}\right)^2+\frac{5}{2}=\frac{53}{4}$

New equation:

$$x^2-7x+\frac{53}{4}=0\Rightarrow 4x^2-28x+53=0$$

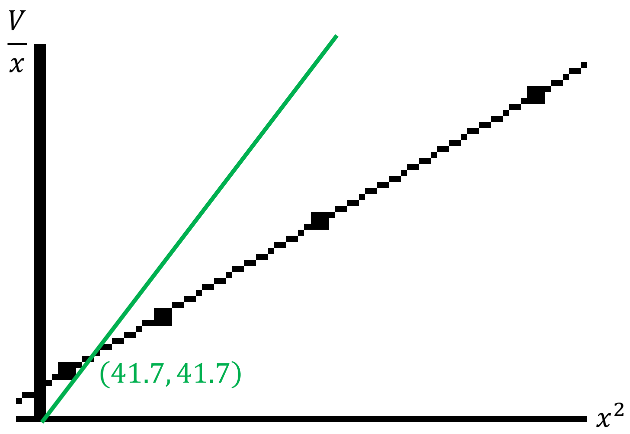
Question 11

(i) $V = x(px^2 + q)$

$\frac{V}{x} = px^2 + q$

x^2	25	100	225	400
$\frac{V}{x}$	35	65	115	185

Plotting on a graph paper:



p = gradient of line $= 0.40$
q = vertical-intercept $= 25$

(ii) When the cuboid is a cube,

$V = x^3$

$x(px^2 + q) = x^3$

$x(0.4x^2 + 25) = x^3$

Since $x \neq 0$,

$0.4x^2 + 25 = x^2$

$x^2 = \frac{125}{3} \Rightarrow x = \sqrt{\frac{125}{3}} = 6.45$

(iii) When the cuboid is a cube,

$V = x^3 \Rightarrow \frac{V}{x} = x^2$

$\therefore$ By drawing $\frac{V}{x} = x^2$ onto the graph and finding the coordinates of intersection of the two lines will give an estimate of x^2 when the cuboid is a cube. And hence by finding the square root of x^2 will estimate the value of x when this happens.

From the graph,

coordinates of intersection $= (41.7, 41.7)$

$\therefore x^2 = 41.7$

$x = 6.46$, which is similar to the answer found in (ii). (verified)