O-LEVEL A-MATHS 2015 – PAPER 1

Question 1

$f(x) = x^2(1-x)$
$\quad = x^2 - x^3$

$f'(x) = 2x - 3x^2$

For f to be an increasing function,
let $f'(x) \geq 0$
$2x - 3x^2 \geq 0$
$3x^2 - 2x \leq 0$
$x(3x-2) \leq 0$
$0 \leq x \leq \frac{2}{3}$

Question 2

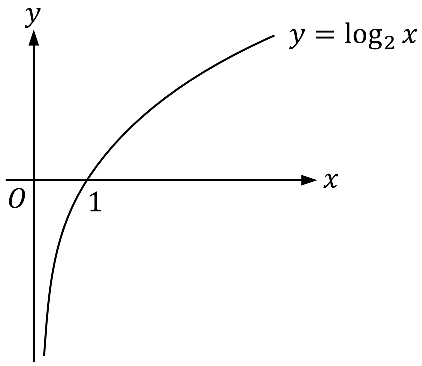
(i) At $(8, 3)$,
$\log_a 8 = 3$
$a^3 = 8 \Rightarrow a = 2$

$\therefore y = \log_2 x$

At $(1, b)$,
$b = \log_2 1 = 0$

At $(c, -2)$,
$\log_2 c = -2$
$c = 2^{-2} = \frac{1}{4}$

(ii) $y = \log_2 x$



Question 3

$N = N_0 e^{kt}$

Let $t = T$ when $N = N_0$

$N_0 e^{kT} = N_0$

$e^{kT} = 1$

When $t = T + 3$,

$N = 2N_0$

$N_0 e^{k(T+3)} = 2N_0$

$e^{k(T+3)} = 2$

$e^{kT} e^{3k} = 2$

$(1) e^{3k} = 2$

$3k = \ln 2 \Rightarrow k = \frac{1}{3} \ln 2$

Question 4

(i) For $ax^2 + 6x + c$ to be always negative,

$a < 0$ and Discriminant < 0

$6^2 - 4ac < 0$

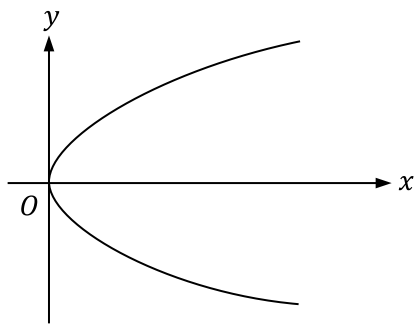
$4ac > 36$

$ac > 9$

(ii) $a = -1$, $c = -10$

Question 5

(i) $y^2 = 4x$



(ii) $y = x - 1$ (1)
$y^2 = 4x$ (2)

Sub. (1) into (2)

$$(x-1)^2 = 4x$$
$$x^2 - 2x + 1 = 4x$$
$$x^2 - 6x + 1 = 0$$
$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(1)}}{2(1)}$$
$$= \frac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}$$

Sub. $x = 3 - 2\sqrt{2}$ into (1),
$$y = 3 - 2\sqrt{2} - 1 = 2 - 2\sqrt{2}$$

$A\left(3 - 2\sqrt{2}, 2 - 2\sqrt{2}\right)$

Sub. $x = 3 + 2\sqrt{2}$ into (1),
$$y = 3 + 2\sqrt{2} - 1 = 2 + 2\sqrt{2}$$

$B\left(3 + 2\sqrt{2}, 2 + 2\sqrt{2}\right)$

$\therefore$ mid-point $= \left(\frac{3 - 2\sqrt{2} + 3 + 2\sqrt{2}}{2}, \frac{2 - 2\sqrt{2} + 2 + 2\sqrt{2}}{2}\right) = (3, 2)$

For line $x + y = 5$, when $x = 3$ and $y = 2$,
LHS = 3 + 2 = 5 = RHS

$\therefore$ mid-point of AB lies on the line $x + y = 5$ (shown)

Question 6

(i) $f(x)$
$= 4\cos^2 x - 2\sin^2 x$
$= 2(2\cos^2 x) - (2\sin^2 x)$
$= 2(\cos 2x + 1) - (1 - \cos 2x)$
$= 2\cos 2x + 2 - 1 + \cos 2x$
$= 3\cos 2x + 1$

$a = 3,\ b = 1$

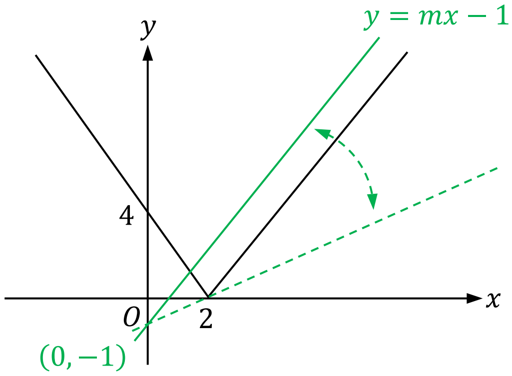
(ii) Greatest $f(x) = 4$
Least $f(x) = -2$

(iii) Period
$= \frac{2\pi}{2} = \pi$

Amplitude $= 3$

Question 7

(i) $y = |2x - 4|$



(ii) When $m = 3$, line: $y = 3x - 1$ (1)
$y = |2x - 4|$ (2)

Sub. (1) into (2),
$3x - 1 = |2x - 4|$

$2x - 4 = -(3x - 1)$
$2x - 4 = -3x + 1$
$5x = 5$
$x = 1$

$y = 3(1) - 1 = 2$

or

$2x - 4 = 3x - 1$
$x = -3$

$y = 3(-3) - 1 = -10$ (NA)

∴ Coordinates of points of intersections are $(1, 2)$.

(iii) From observation,
$\frac{1}{2} < m < 2$

Question 8

(i) $4\tan\theta + 2\cot\theta = 5\sec\theta$

$4\left(\frac{\sin\theta}{\cos\theta}\right) + 2\left(\frac{\cos\theta}{\sin\theta}\right) = 5\left(\frac{1}{\cos\theta}\right)$

$4\sin^2\theta + 2\cos^2\theta = 5\sin\theta$

$4\sin^2\theta + 2(1-\sin^2\theta) - 5\sin\theta = 0$

$4\sin^2\theta + 2 - 2\sin^2\theta - 5\sin\theta = 0$

$2\sin^2\theta - 5\sin\theta + 2 = 0$ (shown)

(ii) $4\tan 2x + 2\cot 2x = 5\sec 2x$

$2\sin^2 2x - 5\sin 2x + 2 = 0$

$(2\sin 2x - 1)(\sin 2x - 2) = 0$

$\sin 2x = \frac{1}{2}$ or $\sin 2x = 2$ (NA)

Basic ∠

$= \sin^{-1}\frac{1}{2} = 30°$

$2x = 30°, 150°, 390°, 510°$

$x = 15°, 75°, 195°, 255°$

Question 9

(i) $\angle DAX = \angle DCA$ (alternate segment theorem)

$\angle DCA = \angle BAC$ (alternate angles, BA // CD)

$\angle BAC = \angle BCA$ (base ∠s of isosceles triangle)

$\therefore \angle DAX = \angle BCA$ (shown)

(ii) $\angle ABC = 180° - 2\angle BCA$ (∠ sum of triangle)

$\angle ABC = 180° - 2\angle DAX$ (from part (i))

$\angle ABC + \angle ADC = 180°$ (∠s in opposite segments)

$\angle ADC = \angle DAX + \angle DXA$ (external ∠ of triangle)

$\therefore \angle ABC + (\angle DAX + \angle DXA) = 180°$

$(180° - 2\angle DAX) + (\angle DAX + \angle DXA) = 180°$

$-\angle DAX + \angle DXA = 0$

$\angle DXA = \angle DAX$

∴ triangle ADX is isosceles. (base ∠s of isosceles triangles) (shown)

Question 10

(i) $a = kt - 2$

$v = \int kt - 2\,dt = \frac{1}{2}kt^2 - 2t + c$

When $t = 0$,

$v = 30$

$\Rightarrow c = 30$

When $t = 20$,

$v = 10$

$\frac{1}{2}k(20)^2 - 2(20) + 30 = 10$

$200k = 20$

$k = 0.1$ (shown)

(ii) $v = \frac{1}{2}(0.1)t^2 - 2t + 30 = \frac{1}{20}t^2 - 2t + 30$

$s = \int \frac{1}{20}t^2 - 2t + 30\,dt$

$= \frac{1}{60}t^3 - t^2 + 30t + c$

When $t = 0$,

$s = c$

When $t = 20$,

$s = \frac{1}{60}(20)^3 - (20)^2 + 30(20) + c = \frac{1000}{3} + c$

Distance

$= \left(\frac{1000}{3} + c\right) - (c) = \frac{1000}{3}\text{ m}$

Question 11

(i) Height of trough $= 20\sin\theta$
$PS = 30 + 2(20\cos\theta) = 30 + 40\cos\theta$

Area, A
$= \frac{1}{2}(20\sin\theta)[30 + (30 + 40\cos\theta)]$
$= 10\sin\theta\,(60 + 40\cos\theta)$
$= 600\sin\theta + 400\sin\theta\cos\theta$
$= 600\sin\theta + 200\sin 2\theta$ (shown)

(ii) $\frac{dA}{d\theta}$
$= 600\cos\theta + 400\cos 2\theta$

Let $\frac{dA}{d\theta} = 0$
$600\cos\theta + 400\cos 2\theta = 0$
$3\cos\theta + 2\cos 2\theta = 0$
$3\cos\theta + 2(2\cos^2\theta - 1) = 0$
$4\cos^2\theta + 3\cos\theta - 2 = 0$
$\cos\theta = \frac{-3 \pm \sqrt{3^2 - 4(4)(-2)}}{2(4)}$
$\cos\theta = \frac{-3 + \sqrt{41}}{8}\ \left(\because 0 < \theta < \frac{\pi}{2}\right)$
$\theta = 1.13$

θ	$(1.1314)^-$	1.1314	$(1.1314)^+$
$\frac{dA}{d\theta}$	$+$	0	$-$
Shape	/	__	\

∴ The trough will hold the maximum amount of water when $\theta = 1.13$.

Question 12

(i) $y = 36(2x+1)^{-2}$

$$\frac{dy}{dx} = -72(2x+1)^{-3}(2) = -\frac{144}{(2x+1)^3}$$

When $\frac{dy}{dt} = -0.36$,

$$\frac{dx}{dt} = 0.02$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \Rightarrow -0.36 = \left(-\frac{144}{(2x+1)^3}\right)(0.02)$$

$$(2x+1)^3 = 8$$

$$2x+1 = 2 \Rightarrow x = \frac{1}{2}$$

(ii) Area of A = Area of B

$$\int_1^a 36(2x+1)^{-2}\,dx = \int_a^4 36(2x+1)^{-2}\,dx$$

$$\int_1^a (2x+1)^{-2}\,dx = \int_a^4 (2x+1)^{-2}\,dx$$

$$\left[\frac{(2x+1)^{-1}}{(2)(-1)}\right]_1^a = \left[\frac{(2x+1)^{-1}}{(2)(-1)}\right]_a^4$$

$$\left[\frac{1}{2x+1}\right]_1^a = \left[\frac{1}{2x+1}\right]_a^4$$

$$\frac{1}{2a+1} - \frac{1}{3} = \frac{1}{9} - \frac{1}{2a+1}$$

$$\frac{2}{2a+1} = \frac{4}{9}$$

$$2a+1 = \frac{9}{2}$$

$$a = \frac{7}{4}$$