A-LEVEL H2 MATHS 2014 – PAPER 2

Question 1

[Ans: (i) 2.5 (ii) $x = \frac{2}{27}y^2$]

(i) $\frac{dx}{dt} = 6t$, $\frac{dy}{dt} = 6$

$\frac{dy}{dx} = \frac{6}{6t} = \frac{1}{t}$

Let $\frac{dy}{dx} = 0.4$

$\frac{1}{t} = 0.4 \Rightarrow t = 2.5$

(ii) At P, $t = p \Rightarrow \frac{dy}{dx} = \frac{1}{p}$

Equation of tangent at P:

$y - 6p = \frac{1}{p}(x - 3p^2)$

$y = \frac{1}{p}x + 3p$

$\therefore D(0, 3p)$

Midpoint of PD

$= \left(\frac{3p^2}{2}, \frac{6p + 3p}{2}\right) = \left(\frac{3p^2}{2}, \frac{9p}{2}\right)$

$\therefore x = \frac{3}{2}p^2, \qquad y = \frac{9}{2}p \Rightarrow p = \frac{2}{9}y$

Cartesian equation of locus of midpoint:

$x = \frac{3}{2}\left(\frac{2y}{9}\right)^2 \Rightarrow x = \frac{2}{27}y^2$

Question 2

[Ans: $a=\frac{3}{2}$, $b=\frac{13}{45}$, $c=\frac{8}{3}$, $d=\frac{2}{3}$]

Let $\frac{9x^2+x-13}{(2x-5)(x^2+9)}$

$= \frac{A}{2x-5}+\frac{Bx+C}{x^2+9}$

$9x^2+x-13 = A(x^2+9)+(Bx+C)(2x-5)$

Let $x=\frac{5}{2}$,

$9\left(\frac{5}{2}\right)^2+\frac{5}{2}-13 = A\left[\left(\frac{5}{2}\right)^2+9\right]$

$A=3$

Let $x=0$,

$-13 = 3(9)+C(-5)$

$C=8$

Let $x=1$,

$9+1-13 = 3(10)+(B+8)(-3)$

$B=3$

$\therefore \frac{9x^2+x-13}{(2x-5)(x^2+9)} = \frac{3}{2x-5}+\frac{3x+8}{x^2+9}$

$$\int_0^2 \frac{9x^2+x-13}{(2x-5)(x^2+9)}\,dx$$
$$= \int_0^2 \frac{3}{2x-5}+\frac{3x+8}{x^2+9}\,dx$$
$$= \int_0^2 \frac{3}{2x-5}+\frac{\frac{3}{2}(2x)+8}{x^2+9}\,dx$$
$$= \int_0^2 \frac{3}{2x-5}+\frac{3}{2}\left(\frac{2x}{x^2+9}\right)+8\left(\frac{1}{x^2+3^2}\right)dx$$
$$= \left[\frac{3}{2}\ln|2x-5|+\frac{3}{2}\ln|x^2+9|+\frac{8}{3}\tan^{-1}\frac{x}{3}\right]_0^2$$
$$= \left(\frac{3}{2}\ln 1+\frac{3}{2}\ln 13+\frac{8}{3}\tan^{-1}\frac{2}{3}\right)-\left(\frac{3}{2}\ln 5+\frac{3}{2}\ln 9+\frac{8}{3}\tan^{-1}\frac{0}{3}\right)$$
$$= \frac{3}{2}\ln\frac{13}{(5)(9)}+\frac{8}{3}\tan^{-1}\frac{2}{3}$$
$$= \frac{3}{2}\ln\frac{13}{45}+\frac{8}{3}\tan^{-1}\frac{2}{3}$$

$a=\frac{3}{2}, b=\frac{13}{45}, c=\frac{8}{3}, d=\frac{2}{3}$

Question 3

[Ans: (i)(a) 440 (b) $4n(n+1)$; 35 stages (ii) $8(2^n - 1)$; 1816 m, away from O]

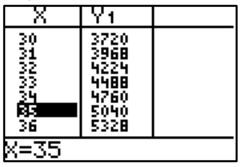
(i) (a)

Stage	Total Distance
1	$2(4)$
2	$2(4+8)$
3	$2(4+8+12)$
⋮	
10	$2\left(\frac{10}{2}\right)[2(4)+(10-1)(4)] = 440$

(b) Total distance after completing n stages

$2\left(\frac{n}{2}\right)[2(4)+(n-1)(4)]$
$= n(4n+4)$
$= 4n(n+1)$

Let $4n(n+1) \geq 5000$
From GC,



X	Y1
30	3720
31	3968
32	4224
33	4488
34	4760
35	5040
36	5328

X=35

The athlete needs to run at least 35 stages to complete at least 5 km.

(ii)

Stage	Total Distance
1	$2(4) = 2(2^2)$
2	$2(4+8) = 2(2^2+2^3)$
3	$2(4+8+16) = 2(2^2+2^3+2^4)$
4	$2(4+8+16+32) = 2(2^2+2^3+2^4+2^5)$
⋮	
n	$2\left[\frac{4[2^n-1]}{2-1}\right] = 8(2^n-1)$

Let $8(2^n - 1) = 10000$
$2^n = 1251$
$n = \frac{\ln 1251}{\ln 2} = 10.289$

Total distance ran from Stage 1 to Stage 10
$= 8(2^{10} - 1) = 8184$

∴ Distance ran after Stage 10 to complete a total distance of 10km
$= 10000 - 8184 = 1816$

Distance to run for only Stage 11
$= 2(2^{12}) = 8192$

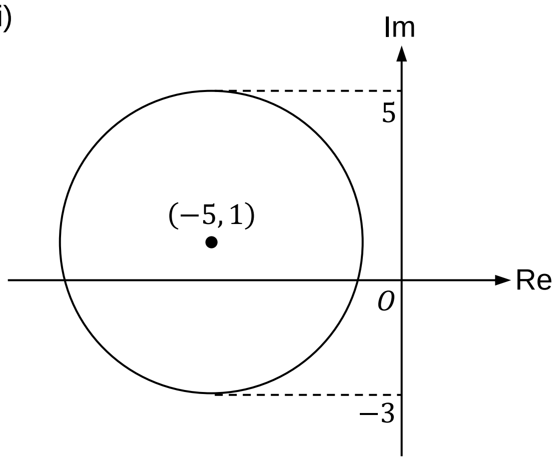
$\therefore OA_{11} = \frac{8192}{2} = 4096 > 1816$

∴ Distance of athlete from O after running exactly 10km is 1816 m and he is running away from O.

Question 4

[Ans: (a)(i) sketch (ii) $z = -2\sqrt{2} - 5 + i(1 + 2\sqrt{2})$ or $2\sqrt{2} - 5 + i(1 - 2\sqrt{2})$ (b)(i) $64e^{i\pi}$ (ii) $5, 11$ and 17]

(a) (i)



(ii) Cartesian equation of $|z + 5 - i| = 4$:

$$(x+5)^2 + (y-1)^2 = 4^2 \quad (1)$$

For $|z - 6i| = |z + 10 + 4i|$:

Let $A(0, 6)$, $B(-10, -4)$

Midpoint of AB

$$= \left(\frac{0-10}{2}, \frac{6-4}{2}\right) = (-5, 1)$$

Gradient of AB

$$= \frac{6+4}{0+10} = 1$$

Cartesian equation:

$$y - 1 = -1(x+5)$$
$$y = -x - 4 \quad (2)$$

Sub. (2) into (1),

$$(x+5)^2 + (-x-4-1)^2 = 16$$
$$2(x+5)^2 = 16$$
$$(x+5)^2 = 8$$
$$x = -5 \pm \sqrt{8} = -5 \pm 2\sqrt{2}$$

When $x = -5 - 2\sqrt{2}$,

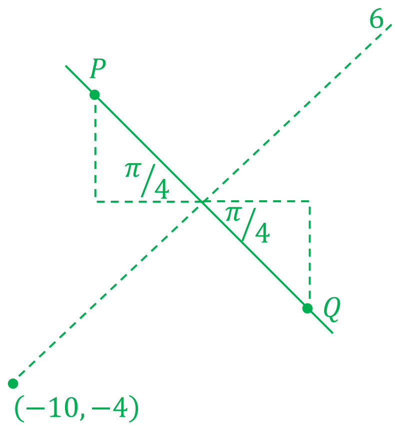
$$y = -(-5 - 2\sqrt{2}) - 4$$
$$= 1 + 2\sqrt{2}$$

When $x = -5 + 2\sqrt{2}$,

$$y = -(-5 + 2\sqrt{2}) - 4$$
$$= 1 - 2\sqrt{2}$$

∴ Possible values of z are $-5 - 2\sqrt{2} + i(1 + 2\sqrt{2})$ and $-5 + 2\sqrt{2} + i(1 - 2\sqrt{2})$.

Alternatively



From observation, at P,

$$z = \left[-5 - 4\left(\cos\frac{\pi}{4}\right)\right] + i\left[1 + 4\left(\sin\frac{\pi}{4}\right)\right]$$
$$= -5 - 2\sqrt{2} + i\left(1 + 2\sqrt{2}\right)$$

From observation, at Q,

$$z = \left[-5 + 4\left(\cos\frac{\pi}{4}\right)\right] + i\left[1 - 4\left(\sin\frac{\pi}{4}\right)\right]$$
$$= -5 + 2\sqrt{2} + i\left(1 - 2\sqrt{2}\right)$$

(b) (i) $|w| = \sqrt{3+1} = 2$

$$\arg w = -\tan^{-1}\frac{1}{\sqrt{3}} = -\frac{\pi}{6}$$

$$\therefore w = 2e^{-i\frac{\pi}{6}}$$
$$w^6 = 2^6 e^{-i\pi} = 64e^{i\pi}$$

(ii) $$\frac{w^n}{w^*} = \frac{2^n e^{-i\frac{n\pi}{6}}}{2e^{i\frac{\pi}{6}}} = 2^{n-1}e^{-i\frac{n\pi}{6}-i\frac{\pi}{6}} = 2^{n-1}e^{-i\frac{\pi}{6}(n+1)}$$

$$= 2^{n-1}\left[\cos\frac{\pi}{6}(n+1) - i\sin\frac{\pi}{6}(n+1)\right]$$

For $\frac{w^n}{w^*}$ to be real,

$$\sin\frac{\pi}{6}(n+1) = 0$$
$$\frac{\pi}{6}(n+1) = k\pi, \qquad k \in \mathbb{Z}$$
$$n = 6k - 1$$
$$n = \cdots, -13, -7, -1, 5, 11, 17, \ldots$$

$\therefore$ Smallest positive whole numbers of n are $5, 11$ and 17.

Question 5

[Ans: (i) describe (ii) evenly spread; biasness]

(i) The marketing manager will be expecting to survey 500 customers ($5\% \times 10000$). In order to carry out a survey using systematic sampling, the manager will have to list the 10000 customers from no. 1 to no. 10000, and choose the customers according to their numberings in intervals of 20s ($\frac{10000}{500}$). Then the manager can start pick any customer within the first 20 and continue collecting their opinions for every other 20^{th} customers. For example, if the manager starts with the 5^{th} customer, then survey can be done with customer no. 25, 45, 65, $\ldots$.

(ii) Advantage: The customers surveyed are more evenly spread out over the 10000 customers.
Disadvantage: There may be biasness due to the periodicity nature of systematic sampling.

Question 6

[Ans: (i) 31500 (ii) 16800 (iii) 8820]

(i) No. of teams
$= {}^3C_1\,{}^8C_4\,{}^5C_2\,{}^6C_4 = 31500$

(ii) No. of teams including only the midfielder brother
$= {}^3C_1\,{}^8C_4\,{}^4C_1\,{}^5C_4 = 4200$

No. of teams including only the attacker brother
$= {}^3C_1\,{}^8C_4\,{}^4C_2\,{}^5C_3 = 12600$

Total no. of teams
$= 4200 + 12600 = 16800$

(iii) No. of teams if one of the midfielders plays as midfielder
$= {}^3C_1\,{}^8C_4\,{}^3C_1\,{}^5C_4 = 3150$

No. of teams if one of the midfielders plays as defender
$= {}^3C_1\,{}^8C_3\,{}^3C_2\,{}^5C_4 = 2520$

No. of teams if one of the midfielders does not play
$= {}^3C_1\,{}^8C_4\,{}^3C_2\,{}^5C_4 = 3150$

Total no. of teams
$= 3150 + 2520 + 3150 = 8820$

Question 7

[Ans: (i) 0.155 (ii) 0.273 (iii) 0.350]

(i) Let X be the no. of 6's shown by the fair die in 10 rolls.

$$X \sim B\left(10, \frac{1}{6}\right)$$

$P(X = 3) = 0.155$

(ii) Let Y be the no. of 6's shown by the fair die in 60 rolls.

$$Y \sim B\left(60, \frac{1}{6}\right)$$

$n = 60$ (large), $np = 60\left(\frac{1}{6}\right) = 10 > 5$, $n(1-p) = 60\left(1 - \frac{1}{6}\right) = 50 > 5$

$$E(Y) = 10, \qquad Var(Y) = 60\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = \frac{25}{3}$$

$$\therefore Y \sim N\left(10, \frac{25}{3}\right) \text{ approx.}$$

$P(5 \le Y \le 8)$
$= P(4.5 < Y < 8.5)$ (c.c.)
$= 0.273$

(iii) Let W be the no. of 6's shown by the biased die in 60 rolls.

$$W \sim B\left(60, \frac{1}{15}\right)$$

$n = 60$ (large), $np = 60\left(\frac{1}{15}\right) = 4 < 5$

$$E(W) = 4$$

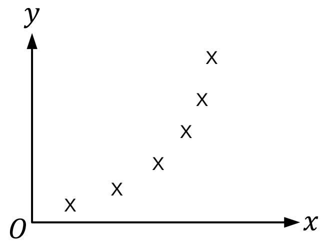
$$\therefore W \sim Po(4) \text{ approx.}$$

$P(5 \le W \le 8) = P(W \le 8) - P(W \le 4)$
$= 0.350$

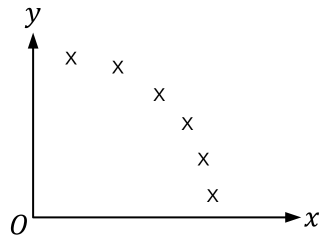
Question 8

[Ans: (a) sketch (b)(i) -0.9470, -0.9749 (ii) $P = -33700 \ln m + 196000$ (iii) $\$64000.00$]

(a) (i) $y = px^2 + t$, $p > 0$, $t > 0$

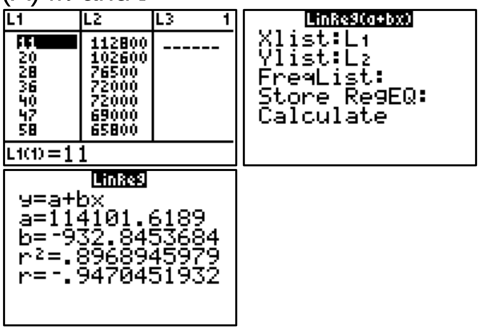


(ii) $y = px^2 + t, p < 0, t > 0$



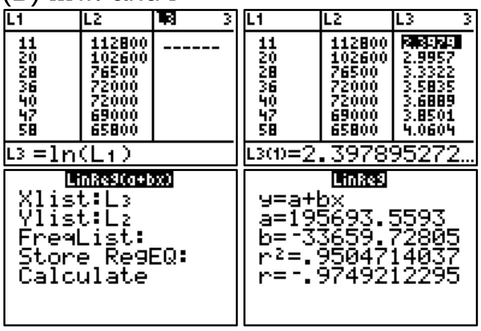
(b) (i) From GC,

(A) m and P



$r = -0.9470$

(B) $\ln m$ and P



$r = -0.9749$

(ii) (B) is a better model as its $|r|$ is closer to 1.
Using (B), the regression line is:
$P = -33700 \ln m + 196000$

(iii) When $m = 50$,

$$P = -33660 \ln 50 + 195694$$
$$= \$64000.00$$

Question 9

[Ans: (i) $H_0: \mu = 4.3$, $H_1: \mu < 4.3$ (ii) $\bar{t} > 3.48$ (iii) $k^2 < 0.423$]

(i) Let the population mean number of minutes the bus is late be μ.

$H_0: \mu = 4.3$

$H_1: \mu < 4.3$

(ii) Let X be the number of minutes that the bus is late.

$n = 10$

$\bar{X} = \bar{t}$

$$s^2 = \frac{10}{10-1}(3.2) = \frac{32}{9}$$

Test Statistics,

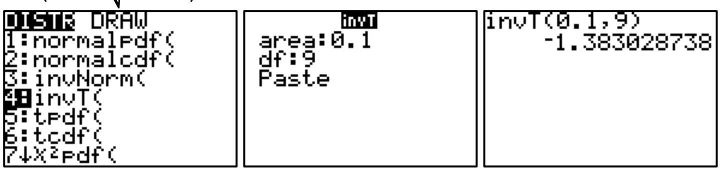
$$T = \frac{\bar{X} - 4.3}{\sqrt{\frac{32/9}{10}}} \sim t(9)$$

Do not reject H_0,

p-value > 0.1

$P(\bar{X} < \bar{t}) > 0.1$

$$P\left(T < \frac{\bar{t} - 4.3}{\sqrt{\frac{32/9}{10}}}\right) > 0.1$$



$$\frac{\bar{t} - 4.3}{\sqrt{\frac{32/9}{10}}} > -1.3830$$

$\bar{t} > 3.48$

(iii) $n = 10$

$\bar{X} = \bar{t} = 4.0$

$$s^2 = \frac{10}{10-1}(k^2) = \frac{10}{9}k^2$$

Test Statistics,

$$T = \frac{\bar{X} - 4.3}{\sqrt{\frac{(10/9)k^2}{10}}} = \frac{\bar{X} - 4.3}{\sqrt{\frac{1}{9}k^2}} \sim t(9)$$

To reject H_0,

p-value < 0.1

$P(\bar{X} < 4.0) < 0.1$

$$P\left(T < \frac{4.0 - 4.3}{\sqrt{\frac{1}{9}k^2}}\right) < 0.1$$

$$\frac{-0.3}{\sqrt{\frac{1}{9}k^2}} < -1.3830$$

$$-1.3830\sqrt{\frac{1}{9}k^2} > -0.3$$

$$\sqrt{\frac{1}{9}k^2} < 0.21692$$

$$\frac{1}{9}k^2 < 0.047054$$

$$k^2 < 0.423$$

Question 10

[Ans: (i)(a) $\frac{1}{500}$ (b) $\frac{44}{125}$ (c) $\frac{29}{500}$ (ii) $\frac{71}{306}$]

(i) (a) Required probability

$$= \left(\frac{1}{10}\right)\left(\frac{2}{10}\right)\left(\frac{1}{10}\right) = \frac{1}{500}$$

(b) Required probability

$$= 1 - \left(\frac{9}{10}\right)\left(\frac{8}{10}\right)\left(\frac{9}{10}\right) = \frac{44}{125}$$

(c) Required probability

$$= P(\times\times+) + P(\times+\times) + P(+\times\times)$$

$$= \left(\frac{3}{10}\right)\left(\frac{1}{10}\right)\left(\frac{2}{10}\right) + \left(\frac{3}{10}\right)\left(\frac{3}{10}\right)\left(\frac{4}{10}\right) + \left(\frac{4}{10}\right)\left(\frac{1}{10}\right)\left(\frac{4}{10}\right)$$

$$= \frac{29}{500}$$

(ii) Event A: two of the symbols are $+$ and o

Event B: exactly one of the symbols is $*$

Required probability

$$= P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(*+o) + P(*\,o+) + P(+o\,*) + P(+*\,o) + P(o+*) + P(o*+)}{P(exactly\ one\ *)}$$

$$= \frac{\left(\frac{1}{10}\right)\left(\frac{3}{10}\right)\left(\frac{3}{10}\right) + \left(\frac{1}{10}\right)\left(\frac{4}{10}\right)\left(\frac{2}{10}\right) + \left(\frac{4}{10}\right)\left(\frac{4}{10}\right)\left(\frac{1}{10}\right) + \left(\frac{4}{10}\right)\left(\frac{2}{10}\right)\left(\frac{3}{10}\right) + \left(\frac{2}{10}\right)\left(\frac{3}{10}\right)\left(\frac{1}{10}\right) + \left(\frac{2}{10}\right)\left(\frac{2}{10}\right)\left(\frac{2}{10}\right)}{\left(\frac{1}{10}\right)\left(\frac{8}{10}\right)\left(\frac{9}{10}\right) + \left(\frac{9}{10}\right)\left(\frac{2}{10}\right)\left(\frac{9}{10}\right) + \left(\frac{9}{10}\right)\left(\frac{8}{10}\right)\left(\frac{1}{10}\right)}$$

$$= \frac{71}{306}$$

Question 11

[Ans: (i)(a) 0.768 (b) 0.675 (ii) $e^{-2n}(1+2n+2n^2)<0.01$, 5 (iii) 0.816 (iv) reasons]

(i) (a) Let X be the no. of prints sold in a randomly chosen week.

$$X \sim Po(11)$$

$P(X>8)=1-P(X\le 8)=0.768$

(b) Let Y be the no. of originals sold in a randomly chosen week.

$$Y \sim Po(2)$$
$$X+Y \sim Po(13)$$

$P(X+Y<15)=P(X\le 14)=0.675$

(ii) Let W be the no. of originals sold in n weeks.

$$W \sim Po(2n)$$

$P(W<3)<0.01$

$P(W=0)+P(W=1)+P(W=2)<0.01$

$e^{-2n}+e^{-2n}\left(\frac{2n}{1!}\right)+e^{-2n}\left(\frac{4n^2}{2!}\right)<0.01$

$e^{-2n}(1+2n+2n^2)<0.01$

From GC,

X	Y1	
1	.67668	
2	.2381	
3	.06197	
4	.01375	
5	.00277	
6	5.2E-4	
7	9.4E-5	

X=5

Smallest $n=5$

(iii) Let T be the no. of prints sold in a year (52 weeks).

$$T \sim Po(572)$$

$\lambda = 572 > 10$

$$\therefore T \sim N(572, 572) \text{ approx.}$$

$P(T>550)$
$=P(T\ge 551)$
$=P(T>550.5)$ (c.c.)
$=0.816$

(iv) Firstly, it may not be valid to assume that the distributions can be modelled as Poisson Distributions because the demand for both original and prints may fluctuate based on perceptions on their value by the buyers at different period.

Secondly, the purchase of originals and prints may not be independent of each other as, for instance, a price comparison between the two may influence the buyers in choosing one over the other to purchase.