‘A’ LEVEL H2 MATHS 2013 – PAPER 2

Question 1

[Ans: (i) explain (ii) $gf(x) = -3\left(\frac{1+x}{1-x}\right)$; 4]

(i) From graph of $y = g(x)$, $R_g = (-\infty, \infty)$

$(-\infty, \infty) = R_g \not\subseteq D_f = (-\infty, 1) \cup (1, \infty)$

$\therefore fg$ does not exist.

(ii) $gf(x) = 1 - 2\left(\frac{2+x}{1-x}\right)$

$gf(x) = \frac{1 - x - 4 - 2x}{1 - x}$

$gf(x) = -3\left(\frac{1+x}{1-x}\right)$

Let $(gf)^{-1}(5) = k$

$\therefore gf(k) = 5$

$-3\left(\frac{1+k}{1-k}\right) = 5$

$\frac{1+k}{1-k} = -\frac{5}{3}$

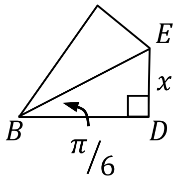
$3 + 3k = -5 + 5k$

$k = 4 \Rightarrow (gf)^{-1}(5) = 4$

Question 2

[Ans: (i) show (ii) $\frac{1}{54}a^3$]

(i)



$\tan\frac{\pi}{6} = \frac{x}{BD} \Rightarrow BD = x\sqrt{3}$

∴ For the triangular base of the prism, length of side $= a - 2\,x\sqrt{3}$

Area of triangular base of the prism

$= \frac{1}{2}\left(a - 2\,x\sqrt{3}\right)\left(a - 2\,x\sqrt{3}\right)\sin\frac{\pi}{3}$

$= \frac{1}{2}\left(a - 2\,x\sqrt{3}\right)^2\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{4}\sqrt{3}\left(a - 2\,x\sqrt{3}\right)^2$

$\therefore V = \left[\frac{1}{4}\sqrt{3}\left(a - 2\,x\sqrt{3}\right)^2\right]x$

$= \frac{1}{4}x\sqrt{3}\left(a - 2\,x\sqrt{3}\right)^2$

(shown)

(ii) From (i),

$V = \frac{1}{4}x\sqrt{3}\left(a^2 - 4ax\sqrt{3} + 12x^2\right)$

$= \frac{1}{4}\sqrt{3}\left(a^2x - 4ax^2\sqrt{3} + 12x^3\right)$

$\frac{dV}{dx} = \frac{1}{4}\sqrt{3}\left(a^2 - 8ax\sqrt{3} + 36x^2\right)$

Let $\frac{dV}{dx} = 0$

$\frac{1}{4}\sqrt{3}\left(a^2 - 8ax\sqrt{3} + 36x^2\right) = 0$

$a^2 - 8ax\sqrt{3} + 36x^2 = 0$

$x = \frac{8a\sqrt{3} \pm \sqrt{192a^2 - 144a^2}}{72} = \frac{8a\sqrt{3} \pm 4a\sqrt{3}}{72}$

$x = \frac{1}{6}a\sqrt{3}$ or $\frac{1}{18}a\sqrt{3}$

$$\frac{d^2V}{dx^2} = \frac{1}{4}\sqrt{3}\left(-8a\sqrt{3} + 72x\right)$$

When $x = \frac{1}{6}a\sqrt{3}$,

$$\frac{d^2V}{dx^2} = \frac{1}{4}\sqrt{3}\left[-8a\sqrt{3} + 72\left(\frac{1}{6}a\sqrt{3}\right)\right]$$
$$= \frac{1}{4}\sqrt{3}\left(4a\sqrt{3}\right) > 0$$

$\therefore V$ is minimum.

When $x = \frac{1}{18}a\sqrt{3}$,

$$\frac{d^2V}{dx^2} = \frac{1}{4}\sqrt{3}\left[-8a\sqrt{3} + 72\left(\frac{1}{18}a\sqrt{3}\right)\right]$$
$$= \frac{1}{4}\sqrt{3}\left(-4a\sqrt{3}\right) < 0$$

$\therefore V$ is maximum.

$\therefore$ max. V

$$= \frac{1}{4}\left(\frac{1}{18}a\sqrt{3}\right)\sqrt{3}\left[a - 2\left(\frac{1}{18}a\sqrt{3}\right)\sqrt{3}\right]^2$$
$$= \frac{1}{24}a\left(\frac{2}{3}a\right)^2 = \frac{1}{54}a^3$$

Question 3

[Ans: (i) $f(0)=0$, $f'(0)=2$, $f''(0)=-4$, $f'''(0)=14$; $f(x)=2x-2x^2+\frac{7}{3}x^3+\cdots$ (ii) $a=-1$, $n=2$; $-\frac{1}{3}x^3$]

(i) Let $f(x)=y=\ln(1+2\sin x)$

$e^y=1+2\sin x$

$$e^y\frac{dy}{dx}=2\cos x$$

$$e^y\frac{d^2y}{dx^2}+e^y\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)=-2\sin x$$

$$e^y\frac{d^2y}{dx^2}+e^y\left(\frac{dy}{dx}\right)^2=-2\sin x$$

$$e^y\frac{d^3y}{dx^3}+e^y\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+e^y(2)\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+e^y\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)^2=-2\cos x$$

$$e^y\frac{d^3y}{dx^3}+3e^y\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+e^y\left(\frac{dy}{dx}\right)^3=-2\cos x$$

When $x=0$,

$y=\ln(1+0)=0=f(0)$

$$\frac{dy}{dx}=\frac{2(1)}{1}=2=f'(0)$$

$$\frac{d^2y}{dx^2}=\frac{-2(0)-(1)(2)^2}{1}$$

$$=-4=f''(0)$$

$$\frac{d^3y}{dx^3}=\frac{-2(1)-3(1)(2)(-4)-(1)(2)^3}{1}$$

$$=14=f'''(0)$$

$$\therefore f(x)=0+2x+\frac{(-4)}{2!}x^2+\frac{14}{3!}x^3+\cdots$$

$$=2x-2x^2+\frac{7}{3}x^3+\cdots$$

(ii) $e^{ax}\sin nx$

$$=\left[1+ax+\frac{(ax)^2}{2!}\ldots\right]\left[nx-\frac{(nx)^3}{3!}+\cdots\right]$$

$$=nx-\frac{n^3x^3}{6}+anx^2+\frac{a^2nx^3}{2}+\cdots$$

$$=nx+anx^2+\left(\frac{a^2n}{2}-\frac{n^3}{6}\right)x^3+\cdots$$

$\therefore n=2, a=-1$

Third non-zero term of $e^{ax}\sin nx$

$$=\left[\frac{(-1)^2 2}{2}-\frac{2^3}{6}\right]x^3=-\frac{1}{3}x^3$$

Question 4

[Ans: (i) $40.4°$ (ii) $l\colon \boldsymbol{r}=\begin{pmatrix}-1/6\\-2/3\\0\end{pmatrix}+\lambda\begin{pmatrix}7\\10\\6\end{pmatrix}$ (iii) $c=-49$ or $\frac{35}{13}$]

Given $p_1\colon \boldsymbol{r}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}=1$, $p_2\colon \boldsymbol{r}\cdot\begin{pmatrix}-6\\3\\2\end{pmatrix}=-1$

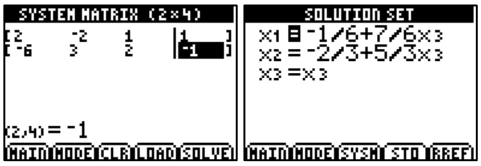
(i) Let the acute angle required be θ.

$$\theta=\cos^{-1}\left|\frac{\begin{pmatrix}2\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}-6\\3\\2\end{pmatrix}}{\left|\begin{pmatrix}2\\-2\\1\end{pmatrix}\right|\left|\begin{pmatrix}-6\\3\\2\end{pmatrix}\right|}\right|=\cos^{-1}\left|\frac{-16}{\sqrt{9}\sqrt{49}}\right|=40.4°$$

(ii) $p_1\colon 2x-2x+z=1$

$p_2\colon -6x+3y+2z=-1$

From GC,



$$l\colon \boldsymbol{r}=\begin{pmatrix}-1/6\\-2/3\\0\end{pmatrix}+\lambda\begin{pmatrix}7\\10\\6\end{pmatrix}$$

(iii) Distance of A to p_1

$$=\left|\frac{\begin{pmatrix}4\\3\\c\end{pmatrix}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}}{\left|\begin{pmatrix}2\\-2\\1\end{pmatrix}\right|}-\frac{1}{\left|\begin{pmatrix}2\\-2\\1\end{pmatrix}\right|}\right|=\left|\frac{8-6+c}{\sqrt{2^2+2^2+1^2}}-\frac{1}{\sqrt{2^2+2^2+1^2}}\right|=\left|\frac{c+1}{3}\right|$$

Distance of A to p_2

$$=\left|\frac{\begin{pmatrix}4\\3\\c\end{pmatrix}\cdot\begin{pmatrix}-6\\3\\2\end{pmatrix}}{\left|\begin{pmatrix}-6\\3\\2\end{pmatrix}\right|}-\frac{-1}{\left|\begin{pmatrix}-6\\3\\2\end{pmatrix}\right|}\right|=\left|\frac{-24+9+2c}{\sqrt{6^2+3^2+2^2}}+\frac{1}{\sqrt{6^2+3^2+2^2}}\right|=\left|\frac{2c-14}{7}\right|$$

$$\left|\frac{c+1}{3}\right|=\left|\frac{2c-14}{7}\right|$$

$\frac{c+1}{3}=\frac{2c-14}{7}$ or $\frac{c+1}{3}=-\frac{2c-14}{7}$

$7c+7=6c-42$ | $7c+7=42-6c$

$c=-49$ | $c=\frac{35}{13}$

Question 5

[Ans: (i) explain (ii) stratified sampling; explain]

(i) To carry out random sampling to choose the 90 employees, a list of all the employees' names can be keyed into a computer's database and allow the computer to randomly select 90 of the entries.
This may not provide the representative sample because the Chief Executive will probably want to involve employees from different departments. But by carrying out random sampling, some employees from certain departments may be missed out.

(ii) A more appropriate sampling method will be stratified sampling. This can be done by randomly selecting employees from each of the company's department, with number chosen from each department being proportional to the relative size of the department.

Question 6

[Ans: $\mu = 1.29a$; $k = 1.29$]

$Y \sim N(\mu, \sigma^2)$

$P(Y < 2a) = 0.95$

$P\left(Z < \dfrac{2a-\mu}{\sigma}\right) = 0.95$

$\dfrac{2a-\mu}{\sigma} = 1.6449 \Rightarrow \sigma = \dfrac{2a-\mu}{1.6449} \quad (1)$

$P(Y < a) = 0.25$

$P\left(Z < \dfrac{a-\mu}{\sigma}\right) = 0.25$

$\dfrac{a-\mu}{\sigma} = -0.67449 \Rightarrow \sigma = \dfrac{\mu-a}{0.67449} \quad (2)$

$(1) = (2)$

$$\frac{2a-\mu}{1.6449} = \frac{\mu-a}{0.67449}$$
$$2a-\mu = 2.4387(\mu-a)$$
$$3.4387\mu = 4.4387a$$
$$\mu = 1.29a$$

$\therefore k = 1.29$

Question 7

[Ans: (i) assumptions (ii) 0.377 (iii) 0.185]

(i) Assumptions:

1. The probability of a packet containing free gift is $\frac{1}{20}$ for each of the packets.

2. Picking a free gift in one packet is independent of doing so for another packet.

(ii) $F \sim B\left(20, \frac{1}{20}\right)$

$P(F = 1) = 0.377$

(iii) $F \sim B\left(60, \frac{1}{20}\right)$

$n = 60$ (large), $np = 3 < 5$

$E(F) = np = 3$

$\therefore F \sim Po(3)$ approx.

$P(F \geq 5) = 1 - P(F \leq 4) = 0.185$

Question 8

[Ans: (i) 0.24 (ii) 0.06 (iii) 0.26]

Given

$$P(A) = 0.7, \qquad P(B|A') = \frac{P(B \cap A')}{P(A')} = 0.8, \qquad P(A|B') = \frac{P(A \cap B')}{P(B')} = 0.88$$

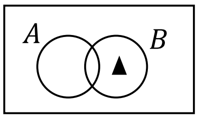
(i) $P(B \cap A')$

$= 0.8\, P(A') = 0.8[1 - P(A)]$

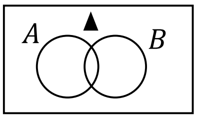
$= 0.8(1 - 0.7) = 0.24$

(ii)

$P(B \cap A')$:



$P(A' \cap B')$:

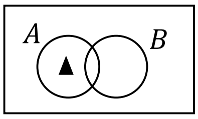


$P(A' \cap B') = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B \cap A')]$

$= 1 - (0.7 + 0.24) = 0.06$

(iii) $P(A \cap B')$:



$$\frac{P(A \cap B')}{P(B')} = 0.88$$

$$\frac{P(A \cap B')}{P(A' \cap B') + P(A \cap B')} = 0.88$$

$$\frac{P(A \cap B')}{0.06 + P(A \cap B')} = 0.88$$

$$P(A \cap B') = 0.0528 + 0.88P(A \cap B')$$

$$P(A \cap B') = \frac{0.0528}{1 - 0.88} = 0.44$$

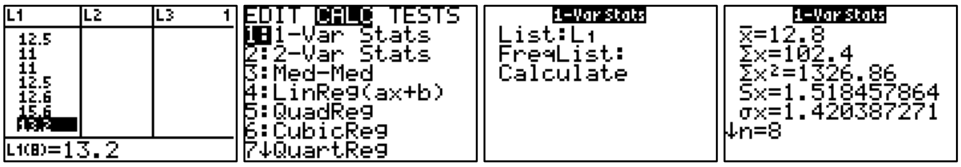
$$P(A) = P(A \cap B') + P(A \cap B)$$

$$P(A \cap B) = 0.7 - 0.44 = 0.26$$

Question 9

[Ans: (i) 12.8 (ii) 2.31 (ii) p-value $= 0.0524$, insufficient evidence]

(i) Let the distance travelled be X.
From GC,



$\bar{x} = 12.8, \qquad s^2 = 1.51846^2 = 2.31$

(ii) Let the (population) mean of the distance travelled be μ.

Assumption: The population variance is unknown.

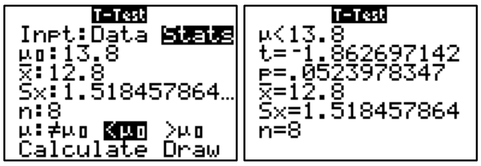
$H_0: \mu = 13.8$
$H_1: \mu < 13.8$

$n = 8$
$\bar{x} = 12.8$
$s^2 = 2.30572$

Test statistics,

$$T = \frac{\overline{X} - 13.8}{\sqrt{2.30571/8}} \sim t(7)$$

From GC,



p-value $= 0.0524$

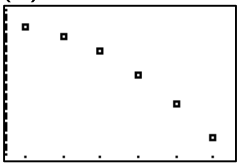
Since p-value > 0.05, there is insufficient evidence for the magazine editor to believe that the figures quoted by car manufacturers for distances travelled per litre of fuel are too high at 5% level of significance.

Question 10

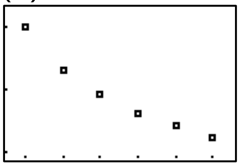
[Ans: (i) sketch (ii) sketch (iii) explain, $r = -0.939$ (iv) $y = 190 - 0.00462x^2$, 134]

(i)

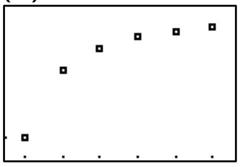
(A)



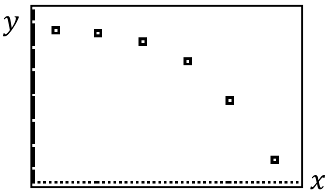
(B)



(C)

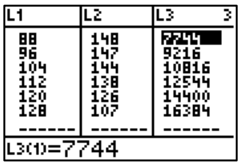


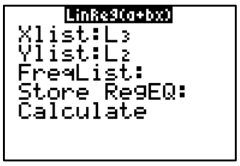
(ii) From GC.

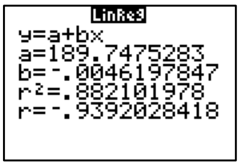


(iii) From the similarity of the scatter diagrams, Case (A) from part (i) will be the most appropriate.

From GC,







$r = -0.939$

(iv) Regression line:

$y = 190 - 0.00462x^2$

When $x = 110$,

distance travelled

$= 189.748 - 0.0046198(110)^2 = 134$

Question 11

[Ans: (i) $\frac{400}{507}$ (ii) $\frac{4}{9}$ (iii) $\frac{1201}{6084}$ (iv) $\frac{1225}{6591}$]

(i) Required probability

$$=\frac{\binom{26}{3}3!\binom{9}{2}2!}{26^3 9^2}=\frac{400}{507}$$

(ii) Required probability

$$=\frac{8+7+6+5+4+3+2+1}{9^2}=\frac{4}{9}$$

(iii) Required probability

$$=P(\text{exactly 2 letters same, digits different})+P(\text{not exactly two letters same, digits same})$$

$$=\left[\binom{26}{2}(2!)\frac{3!}{2!}\left(\frac{1}{26}\right)^3\right]\left[\binom{9}{2}2!\left(\frac{1}{9}\right)^2\right]+\left[\binom{26}{3}3!\left(\frac{1}{26}\right)^3+\binom{26}{1}\frac{3!}{3!}\left(\frac{1}{26}\right)^3\right]\left[\binom{9}{1}\frac{2!}{2!}\left(\frac{1}{9}\right)^2\right]$$

$$=\frac{1201}{6084}$$

(iv) Required probability

$$=P(\text{letters repeat})+P(\text{letters do not repeat})$$

$$=\left[\binom{5}{1}\binom{21}{1}\frac{3!}{2!}\left(\frac{1}{26}\right)^3\right]\left[\binom{5}{1}\binom{4}{1}2!\left(\frac{1}{9}\right)^2\right]+\left[\binom{5}{1}\binom{21}{2}3!\left(\frac{1}{26}\right)^3\right]\left[\binom{5}{1}\binom{4}{1}(2!)2!\left(\frac{1}{9}\right)^2\right]$$

$$=\frac{1225}{6591}$$

Question 12

[Ans: (i) conditions (ii) 4 (iii) 0.397 (iv) 0.848]

(i) Conditions

1. The average number of employees absent must be the same each day. This condition may not be met as some illness can affect more people during different time of the year.
2. The numbers of absences in a day must be independent of another day. Similarly, this condition may not be met because if the illness spread, the numbers of absences the next day may be affected by the current day.

(ii) Let X be the number of employees absent for n number of days from the Administration Department.

$$X \sim Po(1.2n)$$

$P(X = 0) < 0.01$

From GC,

X	Y1	
0	ERROR	
1	.30119	
2	.09072	
3	.02732	
4	.00823	
5	.00248	
6	7.5E-4	

X=4

Smallest number of days $= 4$

(iii) Let Y be the number of employees absent for a 5-day period from the two departments.

$$Y \sim Po\big(5(1.2 + 2.7)\big) \Rightarrow Y \sim Po(19.5)$$

Required probability
$= P(Y > 20)$
$= 1 - P(Y \leq 20) = 0.397$

(iv) Let W be the number of employees absent for a 60-day period from the two departments.

$$W \sim Po\big(60(1.2 + 2.7)\big) \Rightarrow W \sim Po(234)$$

$\lambda = 234 > 10$
$E(W) = 234, \qquad Var(W) = 234$

$$\therefore W \sim N(234, 234) \text{ approx.}$$

Required probability
$= P(200 \leq W \leq 250)$
$= P(199.5 < W < 250.5) \ (c.c)$
$= 0.848$