O-LEVEL A-MATHS 2013 – PAPER 2

Question 1

Given $\boldsymbol{A} = \begin{pmatrix} 2 & -2 \\ 1 & 4 \end{pmatrix}$

$$\boldsymbol{A}^{-1} = \frac{1}{(2)(4)-(1)(-2)}\begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{10} & \frac{1}{5} \end{pmatrix}$$

$$\frac{1}{2}y + 2 = \frac{1}{2}x$$
$$2y + 8 = 2x \Rightarrow 2x - 2y = 8 \quad (1)$$

$$y = -\frac{1}{4} - \frac{1}{4}x$$
$$4y = -1 - x \Rightarrow x + 4y = -1 \quad (2)$$

To solve (1) & (2),
$$\begin{pmatrix} 2 & -2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{10} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} 8 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\therefore \begin{cases} x = 3 \\ y = -1 \end{cases}$$

Question 2

$\angle BAE = \angle CBD$ (alternate segment theorem)
$\angle BEC$
$= \angle BAE + \angle ABE$ (exterior $\angle$ of Δ)
$= \angle CBD + \angle DBE$ ($\angle BAE = \angle CBD$, $\angle ABE = \angle DBE$)
$= \angle CBE$

$\therefore$ triangle BCE is isosceles (base $\angle$s of isos. Δs)

Question 3

(i) $f(-3) = 0$

$(-3)^3 + a(-3) + b = 0$

$-3a + b = 27 \quad (1)$

$f(4) = 56$

$(4)^3 + 4a + b = 56$

$4a + b = -8 \quad (2)$

$(2) - (1)$

$7a = -35 \Rightarrow a = -5$

Sub. $a = -5$ into (2),

$4(-5) + b = -8 \Rightarrow b = 12$

(ii) Given $(x + 3)$ is a factor of $f(x)$.

$$\begin{array}{r|l} & x^2 - 3x + 4 \\ \hline x + 3 & x^3 + 0x^2 - 5x + 12 \\ & \underline{-(x^3 + 3x^2)} \\ & -3x^2 - 5x \\ & \underline{-(-3x^2 - 9x)} \\ & 4x + 12 \\ & \underline{-(4x + 12)} \\ & 0 \end{array}$$

$\therefore f(x) = (x + 3)(x^2 - 3x + 4)$

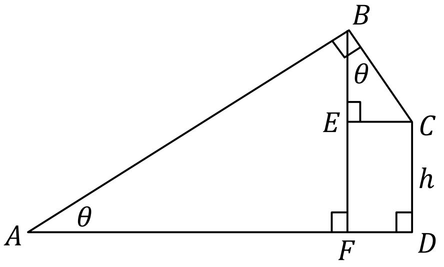
For $x^2 - 3x + 4$,

Discriminant = $(-3)^2 - 4(1)(4) = -7$

$\therefore$ it has no real roots.

$\therefore f(x) = 0$ has 1 real root.

Question 4



(i) $\sin\theta = \dfrac{BF}{AB} \Rightarrow BF = AB\sin\theta = 5\sin\theta$

$\cos\theta = \dfrac{BE}{BC} \Rightarrow BE = BC\cos\theta = \cos\theta$

$\therefore h = BF - BE = 5\sin\theta - \cos\theta$

(ii) Let $5\sin\theta - \cos\theta$

$= R\sin(\theta - \alpha)$

$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

$$\begin{cases} R\cos\alpha = 5 & (1) \\ R\sin\alpha = 1 & (2) \end{cases}$$

$(1)^2 + (2)^2 \quad R^2 = 5^2 + 1^2 \Rightarrow R = \sqrt{26}$

$(2)/(1) \quad \tan\alpha = \dfrac{1}{5} \Rightarrow \alpha = 11.3099°$

$\therefore h = \sqrt{26}\sin(\theta - 11.3°)$

(iii) $h = 3$

$\sqrt{26}\sin(\theta - 11.3099°) = 3$

$\theta - 11.3099° = \sin^{-1}\dfrac{3}{\sqrt{26}}$

$\theta = 47.3°$

Question 5

(i) $3\cos^2 x - \sin^2 x$
$= 2\cos^2 x + (\cos^2 x - \sin^2 x)$
$= (\cos 2x + 1) + \cos 2x$
$= 1 + 2\cos 2x$

$\therefore a = 1,\ b = 2$

(ii) $\int (3\cos^2 x - \sin^2 x)\,dx$
$= \int 1 + 2\cos 2x\,dx$
$= x + 2\left(\frac{\sin 2x}{2}\right) + c$
$= x + \sin 2x + c$

$\int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} (3\cos^2 x - \sin^2 x)\,dx$
$= [x + \sin 2x]_{-\frac{\pi}{12}}^{\frac{\pi}{12}}$
$= \left(\frac{\pi}{12} + \sin\frac{\pi}{6}\right) - \left[-\frac{\pi}{12} + \sin\left(-\frac{\pi}{6}\right)\right]$
$= \frac{\pi}{6} + 1$

Question 6

(i) Period $= \dfrac{360°}{3} = 120°$

(ii) Amplitude $= \dfrac{6-(-2)}{2} = 4$

(iii) From observation,

$q = 4$, $p = 2$

(iv) $y = 2 + 4\sin 3x$

Let $y = 0$

$2 + 4\sin 3x = 0$

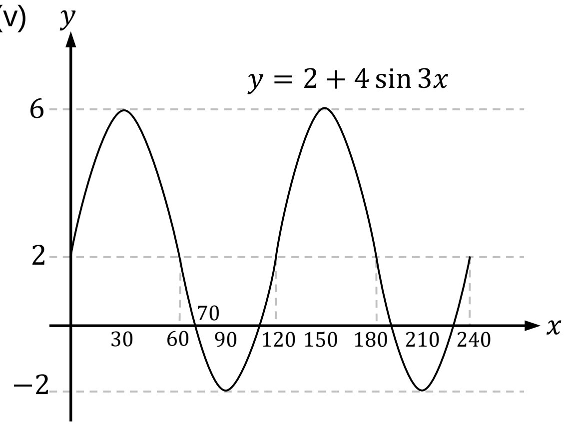
$\sin 3x = -\dfrac{1}{2}$

Basic $\angle = \sin^{-1}\dfrac{1}{2} = 30°$

For the smallest positive value of x,

$3x = 180° + 30° \Rightarrow x = 70°$

(v)



Question 7

(i) Total length of tape $= 600$

$4r + 2l + 2(2\pi r) = 600$

$2r + l + 2\pi r = 300 \Rightarrow l = 300 - 2r - 2\pi r$

$V = \pi r^2 l$

$= \pi r^2(300 - 2r - 2\pi r)$

(shown)

(ii) $V = \pi(300r^2 - 2r^3 - 2\pi r^3)$

$\frac{dV}{dr} = \pi(600r - 6r^2 - 6\pi r^2)$

Let $\frac{dV}{dr} = 0$

$\pi(600r - 6r^2 - 6\pi r^2) = 0$

$6r(100 - r - \pi r) = 0$

$100 - r - \pi r = 0 \; (\because r \neq 0)$

$r + \pi r = 100$

$r = \frac{100}{1+\pi}$ (shown)

$\therefore k = 100$

When $r = \frac{100}{1+\pi}$,

$l = 300 - 2\left(\frac{100}{1+\pi}\right) - 2\pi\left(\frac{100}{1+\pi}\right)$

$= 300 - \frac{200}{1+\pi} - \frac{200\pi}{1+\pi} = 100$

Question 8

(a) $\dfrac{\sqrt{6}+\sqrt{5}}{\sqrt{15}-\sqrt{2}}$

$= \dfrac{\sqrt{6}+\sqrt{5}}{\sqrt{15}-\sqrt{2}}\left(\dfrac{\sqrt{15}+\sqrt{2}}{\sqrt{15}+\sqrt{2}}\right)$

$= \dfrac{\sqrt{90}+\sqrt{12}+\sqrt{75}+\sqrt{10}}{15-2}$

$= \dfrac{\sqrt{(9)(10)}+\sqrt{(4)(3)}+\sqrt{(25)(3)}+\sqrt{10}}{13}$

$= \dfrac{3\sqrt{10}+2\sqrt{3}+5\sqrt{3}+\sqrt{10}}{13}$

$= \dfrac{4\sqrt{10}+7\sqrt{3}}{13}$

$= \dfrac{4}{13}\sqrt{10}+\dfrac{7}{13}\sqrt{3}$

$\therefore a = \dfrac{4}{13}, b = \dfrac{7}{13}$

(b) $2^{2x} = 2^{2+x} + 21$

$(2^x)^2 - 2^2 2^x - 21 = 0$

$(2^x)^2 - 4(2^x) - 21 = 0$

Let $y = 2^x$

$y^2 - 4y - 21 = 0$

$(y+3)(y-7) = 0$

$y = -3$ or $y = 7$

$2^x = -3$ (NA)

$2^x = 7 \Rightarrow x = \dfrac{\ln 7}{\ln 2}$

$x = 2.81$

Question 9

(i) Given $a = -16e^{-0.5t}$

$v = \int -16e^{-0.5t}\,dt$

$= -16\left(\frac{e^{-0.5t}}{-0.5}\right) + c = 32e^{-0.5t} + c$

When $t = 0$,

$v = 28$

$32e^{-0.5(0)} + c = 28 \Rightarrow c = -4$

$\therefore v = 32e^{-0.5t} - 4$

At instantaneous rest,

$v = 0$

$32e^{-0.5t} - 4 = 0$

$e^{-0.5t} = \frac{4}{32} \Rightarrow -0.5t = \ln\frac{1}{8} = -\ln 8$

$t = 2\ln 8 = 4.16$

(ii) $s = \int (32e^{-0.5t} - 4)\,dt$

$s = 32\left(\frac{e^{-0.5t}}{-0.5}\right) - 4t + c$

$s = -64e^{-0.5t} - 4t + c$

When $t = 0$,

$s = 0$

$-64e^{-0.5(0)} - 4(0) + c = 0$

$c = 64$

$\therefore s = -64e^{-0.5t} - 4t + 64$

At instantaneous rest,

$s = -64e^{-0.5(2\ln 8)} - 4(2\ln 8) + 64$

$= 39.4$

The particle is 39.4m from O when it is at instantaneous rest.

Question 10

(i) $x^2 + y^2 + 6x - 4y - 12 = 0$
$(x+3)^2 - 3^2 + (y-2)^2 - 2^2 - 12 = 0$
$(x+3)^2 + (y-2)^2 = 25$

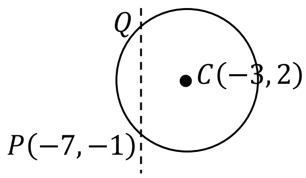
$\therefore$ centre $(-3, 2)$, and radius $= 5$

(ii) Let C be the centre of the circle.

Gradient of CP
$= \dfrac{-1-2}{-7+3} = \dfrac{3}{4}$

Equation of tangent:
$y + 1 = -\dfrac{4}{3}(x+7)$
$3y + 3 = -4x - 28$
$3y + 4x + 31 = 0 \quad (1)$ (shown)

(iii)



From observation, $Q(-7, 5)$.

Gradient of CQ
$= \dfrac{5-2}{-7-(-3)} = -\dfrac{3}{4}$

Equation of tangent at Q:
$y - 5 = \dfrac{4}{3}(x+7)$
$\Rightarrow 3y - 15 = 4x + 28$
$3y - 4x - 43 = 0 \quad (2)$

(iv) $(1) + (2)$
$6y - 12 = 0 \Rightarrow y = 2$

Sub. $y = 2$ into (1)
$3(2) + 4x + 31 = 0 \Rightarrow x = -\dfrac{37}{4}$

$\therefore R\left(-\dfrac{37}{4}, 2\right)$

Question 11

(i) Let $x = 0$,

$$y = 3 - \frac{12}{(0+3)^2} = \frac{5}{3}$$

$\therefore A\left(0, \frac{5}{3}\right)$

Let $y = 0$,

$$3 - \frac{12}{(x+3)^2} = 0$$
$$(x+3)^2 = 4$$
$$x + 3 = \pm 2$$
$$x = -5 \text{ or } x = -1$$

$\therefore B(-1, 0)$ and $C(-5, 0)$

Let $y = \frac{5}{3}$,

$$3 - \frac{12}{(x+3)^2} = \frac{5}{3} \Rightarrow \frac{12}{(x+3)^2} = \frac{4}{3}$$
$$(x+3)^2 = 9$$
$$x + 3 = \pm 3$$
$$x = -6 \text{ or } x = 0$$

$\therefore D\left(-6, \frac{5}{3}\right)$

(ii) Area

$$= \int_{-1}^{0} 3 - \frac{12}{(x+3)^2}\, dx$$
$$= \int_{-1}^{0} 3 - 12(x+3)^{-2}\, dx$$
$$= [3x + 12(x+3)^{-1}]_{-1}^{0}$$
$$= [12(3)^{-1}] - [-3 + 12(-1+3)^{-1}]$$
$$= 1$$

(iii) Area

$$= 6\left(\frac{5}{3}\right) - 1 - \int_{-6}^{-5} 3 - \frac{12}{(x+3)^2}\, dx$$
$$= 9 - [3x + 12(x+3)^{-1}]_{-6}^{-5}$$
$$= 9 - \{[3(-5) + 12(-5+3)^{-1}] - [3(-6) + 12(-6+3)^{-1}]\}$$
$$= 9 - 1$$
$$= 8$$