O-LEVEL A-MATHS 2011 – PAPER 1

Question 1

$\frac{dy}{dx} = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$

For y to be decreasing, let

$\frac{dy}{dx} \leq 0$

$\frac{e^x(x-1)}{x^2} \leq 0$

$x - 1 \leq 0$

$x \leq 1$

∴ When $0 < x \leq 1$ y is decreasing.

Question 2

Given $y = \frac{a}{x+b}$.

$xy + by = a$

$xy = -by + a$

$\therefore -b = -3 \Rightarrow b = 3$

$xy = -3y + a$

At $(2, 6)$,

$6 = -3(2) + a$

$\Rightarrow a = 12$

Question 3

$3^{x+2} = 12^{2-x}$

$3^x 3^2 = 12^2 12^{-x}$

$3^x 12^x = \frac{12^2}{3^2}$

$36^x = 16$

$(6^2)^x = 16$

$(6^x)^2 = 16$

$6^x = -4$ (NA) or $6^x = 4$

Question 4

$2\cos 2\theta = 4 + 5\cos\theta$

$2(2\cos^2\theta - 1) = 4 + 5\cos\theta$

$4\cos^2\theta - 5\cos\theta - 6 = 0$

$(4\cos\theta + 3)(\cos\theta - 2) = 0$

$\cos\theta = -\frac{3}{4}$ or $\cos\theta = 2$ (NA)

Basic ∠ = $\cos^{-1}\frac{3}{4} = 0.72273$

$\theta = \pi - 0.72273, \pi + 0.72273$

$= 2.42, 3.86$

Question 5

$y = k\sqrt{4x+1} = k(4x+1)^{\frac{1}{2}}$

$\frac{dy}{dx} = k\left(\frac{1}{2}\right)(4x+1)^{-\frac{1}{2}}(4)$
$= \frac{2k}{\sqrt{4x+1}}$

When $x = 2$,
$\frac{dy}{dx} = \frac{2k}{\sqrt{4(2)+1}} = \frac{2k}{3}$

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$
$\frac{dy}{dt} = \frac{2k}{3}\left(2\frac{dy}{dt}\right)$
$\frac{4k}{3} = 1$
$k = \frac{3}{4}$

Question 6

Total surface area $= 120$
$2x^2 + 4xy = 120$
$\Rightarrow x^2 + 2xy = 60 \quad (1)$

Total length of edges $= 54$
$8x + 4y = 54$
$4x + 2y = 27$
$\Rightarrow y = \frac{1}{2}(27 - 4x) \quad (2)$

Sub. (2) into (1)
$x^2 + 2x\left[\frac{1}{2}(27 - 4x)\right] = 60$
$x^2 + x(27 - 4x) - 60 = 0$
$-3x^2 + 27x - 60 = 0$
$x^2 - 9x + 20 = 0$ (shown)
$(x - 4)(x - 5) = 0$
$x = 4$ or $x = 5$

When $x = 4$,
$y = \frac{1}{2}(27 - 16) = \frac{11}{2}$

When $x = 5$,
$y = \frac{1}{2}(27 - 20) = \frac{7}{2}$

Question 7

(i) $(2-3x)(1+ax)^6$
$= (2-3x)\left[1 + 6ax + \ {}^6C_2(ax)^2 + \cdots\right]$
$= (2-3x)(1 + 6ax + 15a^2x^2 + \cdots)$
$= 2 + 12ax + 30a^2x^2 - 3x - 18ax^2 + \cdots$
$= 2 + (12a-3)x + (30a^2 - 18a)x^2 + \cdots$

$12a - 3 = 3$
$\Rightarrow a = \frac{1}{2}$

(ii) Coefficient of x^2
$= 30\left(\frac{1}{2}\right)^2 - 18\left(\frac{1}{2}\right)$
$= -\frac{3}{2}$

Question 8

(i) Let $y = 0$
$3 - |2x+1| = 0$
$|2x+1| = 3$
$2x + 1 = 3$ or $2x + 1 = -3$
$x = 1$ $\quad$ $x = -2$
$\therefore A(-2, 0), C(1, 0)$

x-coordinates of B
$= \frac{-2+1}{2} = -\frac{1}{2}$
When $x = -\frac{1}{2}$,
$y = 3 - \left|2\left(-\frac{1}{2}\right) + 1\right| = 3$
$\therefore B\left(-\frac{1}{2}, 3\right)$

(ii) $3 - |2x+1| = x$
$|2x+1| = 3 - x$
$2x + 1 = 3 - x$ or $2x + 1 = -(3-x)$
$3x = 2$ $\quad$ $2x + 1 = -3 + x$
$x = \frac{2}{3}$ $\quad$ $x = -4$

Question 9

For normal to the curve at $x = 1$,

$5y + 2x = 12$

$\Rightarrow y = -\frac{2}{5}x + \frac{12}{5}$

$\therefore$ Gradient of tangent to the curve $= \frac{5}{2}$

For the curve $y = ax^3 + b$,

$\frac{dy}{dx} = 3ax^2$

When $x = 1$,

$\frac{dy}{dx} = \frac{5}{2}$

$3a(1)^2 = \frac{5}{2} \Rightarrow a = \frac{5}{6}$

For normal to the curve at $x = 1$,

$5y + 2(1) = 12$

$5y = 10$

$y = 2$

For the curve,

$y = \frac{5}{6}x^3 + b$

When $x = 1$,

$y = 2$

$\frac{5}{6}(1)^3 + b = 2$

$b = 2 - \frac{5}{6}$

$= \frac{7}{6}$

Question 10

(i) Area $= 9+\sqrt{6}$

$BC\left(2\sqrt{2}+\sqrt{3}\right) = 9+\sqrt{6}$

$$BC = \frac{9+\sqrt{6}}{2\sqrt{2}+\sqrt{3}}\left(\frac{2\sqrt{2}-\sqrt{3}}{2\sqrt{2}-\sqrt{3}}\right)$$
$$= \frac{18\sqrt{2}-9\sqrt{3}+2\sqrt{12}-\sqrt{18}}{\left(2\sqrt{2}\right)^2-\left(\sqrt{3}\right)^2}$$
$$= \frac{18\sqrt{2}-9\sqrt{3}+4\sqrt{3}-3\sqrt{2}}{5}$$
$$= \frac{15\sqrt{2}-5\sqrt{3}}{5}$$
$$= 3\sqrt{2}-\sqrt{3}$$
$$= \sqrt{18}-\sqrt{3}$$

(ii) AC^2

$= AB^2 + BC^2$

$= \left(2\sqrt{2}+\sqrt{3}\right)^2 + \left(\sqrt{18}-\sqrt{3}\right)^2$

$= \left(8+4\sqrt{6}+3\right) + \left(18-2\sqrt{54}+3\right)$

$= 32+4\sqrt{6}-6\sqrt{6}$

$= 32-2\sqrt{6}$

$= 32-\sqrt{24}$

Question 11

(i) Let M be the midpoint of AC.

$M\left(\frac{-2+10}{2}, \frac{2+8}{2}\right) \Rightarrow M(4,5)$

Gradient of AC

$= \frac{8-2}{10+2} = \frac{1}{2}$

Equation of BD:

$y-5 = -2(x-4)$

$y = -2x+13$

$\therefore B(0,13)$

Let $D(a,b)$.

$\frac{a+0}{2} = 4 \Rightarrow a = 8$

$\frac{b+13}{2} = 5 \Rightarrow b = -3$

$\therefore D(8,-3)$

(ii) Area

$= \frac{1}{2}\begin{vmatrix} 0 & -2 & 8 & 10 & 0 \\ 13 & 2 & -3 & 8 & 13 \end{vmatrix}$

$= \frac{1}{2}[(200)-(-40)]$

$= 120$ units^2

Question 12

(i) $s = (20)\ln 21 - 20 = 40.9$

(ii) velocity, $v = \dfrac{ds}{dt}$

$$v = \frac{t}{t+1} + \ln(t+1) - 1$$

When $t = 20$,

$$v = \frac{20}{21} + \ln 21 - 1 = 3.00$$

(iii) acceleration, $a = \dfrac{dv}{dt}$

$$a = \frac{(t+1) - t}{(t+1)^2} + \frac{1}{t+1}$$

$$= \frac{1}{(t+1)^2} + \frac{1}{t+1}$$

When $t = 20$,

$$a = \frac{1}{(21)^2} + \frac{1}{21} = 0.0499$$

Question 13

(i) Let the vertical height of A be h_A.

$\cos\theta = \frac{h_A}{5} \Rightarrow h_A = 5\cos\theta$

Let the vertical height of B from A be h_{AB}.

$\sin\theta = \frac{h_{AB}}{3} \Rightarrow h_{AB} = 3\sin\theta$

$h = h_A + h_{AB} = 5\cos\theta + 3\sin\theta$
$a = 5,\ b = 3$

(ii) Let $h = R\sin(\theta + \alpha)$
$5\cos\theta + 3\sin\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$
$\therefore R\sin\alpha = 5 \quad (1)$
$R\cos\alpha = 3 \quad (2)$
${(1)}/{(2)}\ \tan\alpha = \frac{5}{3} \Rightarrow \alpha = 59.036°$
$(1)^2 + (2)^2\ \ R^2 = 5^2 + 3^2 \Rightarrow R = \sqrt{34}$

$\therefore h = \sqrt{34}\sin(\theta + 59.0°)$

(iii) Max. $h = \sqrt{34}$
This occurs when
$\sin(\theta + 59.036°) = 1$
$\theta + 59.036° = 90°$
$\theta = 31.0°$

(iv) $h = \sqrt{17}$
$\sqrt{34}\sin(\theta + 59.036°) = \sqrt{17}$
$\sin(\theta + 59.036°) = \frac{1}{\sqrt{2}}$
Basic ∠
$= \sin^{-1}\frac{1}{\sqrt{2}} = 45°$

$\theta + 59.036° = 180° - 45° = 135°$
$\theta = 76.0°$