O-LEVEL A-MATHS 2010 – PAPER 2

Question 1

[Ans: $194.5°, 345.5°$]

$3\cot^2\theta + 10\operatorname{cosec}\theta = 5$
$3(\operatorname{cosec}^2\theta - 1) + 10\operatorname{cosec}\theta = 5$
$3\operatorname{cosec}^2\theta + 10\operatorname{cosec}\theta - 8 = 0$
$(3\operatorname{cosec}\theta - 2)(\operatorname{cosec}\theta + 4) = 0$

$\operatorname{cosec}\theta = \frac{2}{3}$ or $\operatorname{cosec}\theta = -4$

$\sin\theta = \frac{3}{2}$ (NA)

$\sin\theta = -\frac{1}{4}$

Basic ∠

$= \sin^{-1}\frac{1}{4} = 14.478°$

$\theta = 180° + 14.478°, 360 - 14.478°$
$\theta = 194.5°, 345.5°$

Question 2

[Ans: (i) shown (ii) $A = 8x - \frac{2x^2}{3}$ (iii) 24]

(i) ΔPWV is similar to ΔPQR

$$\frac{8-y}{8} = \frac{x}{12}$$
$$8 - y = \frac{2x}{3}$$
$$y = 8 - \frac{2x}{3} \text{ (shown)}$$

(ii) $A = xy$
$$= x\left(8 - \frac{2x}{3}\right)$$
$$= 8x - \frac{2x^2}{3}$$

(iii) $\frac{dA}{dx} = 8 - \frac{4x}{3}$

Let $\frac{dA}{dx} = 0$,

$$8 - \frac{4x}{3} = 0 \Rightarrow x = 6$$

$$\frac{d^2A}{dx^2} = -\frac{4}{3} < 0$$

$\therefore$ maximum $A = 8(6) - \frac{2(6)^2}{3} = 24$

Question 3

[Ans: $x=-\frac{1}{4}$ and $y=\frac{5}{4}$ or $x=1$ and $y=-5$]

$32^x \times 2^y = 1$
$(2^5)^x \times 2^y = 1$
$2^{5x} \times 2^y = 1$
$2^{5x+y} = 2^0$
$5x + y = 0$
$y = -5x$ (1)

$3^{x-12} \div 27^y = 81^{\frac{1}{x}}$
$3^{x-12} \div (3^3)^y = 3^{\frac{4}{x}}$
$3^{x-12} \div 3^{3y} = 3^{\frac{4}{x}}$
$3^{x-12-3y} = 3^{\frac{4}{x}}$
$x - 12 - 3y = \frac{4}{x}$
$x^2 - 12x - 3xy = 4$ (2)

Sub (1) into (2),
$x^2 - 12x - 3x(-5x) = 4$
$16x^2 - 12x - 4 = 0 \Rightarrow 4x^2 - 3x - 1 = 0$
$(4x+1)(x-1) = 0$
$x = -\frac{1}{4}$ or $x = 1$
$y = -5\left(-\frac{1}{4}\right) = \frac{5}{4}$
$y = -5(1) = -5$

Question 4

[Ans: (i) $\frac{1}{2}$ (ii) show]

(i) $T_{r+1} = {}^8C_r x^{8-r}\left(-\frac{k}{x^3}\right)^r = {}^8C_r(-k)^r x^{8-4r}$

Let $8 - 4r = 0 \Rightarrow r = 2$

$\therefore\ {}^8C_2(-k)^2 = 7$
$28k^2 = 7$
$k = \frac{1}{2}$

(ii) Let $8 - 4r = -4 \Rightarrow r = 3$

$\therefore$ term with $\frac{1}{x^4}$
$= {}^8C_3\left(-\frac{1}{2}\right)^3 \frac{1}{x^4}$
$= 56\left(-\frac{1}{8}\right)\frac{1}{x^4}$
$= -\frac{7}{x^4}$

$\therefore (1 + x^4)\left(x - \frac{1}{2x^3}\right)^8$
$= (1 + x^4)\left(\ldots + 7 - \frac{7}{x^4} + \cdots\right)$
$= 7 - 7 + \cdots = 0 + \cdots$

$\therefore$ there is no constant term in the expansion. (shown)

Question 5

[Ans: (i) -4 (ii) $y = x - 4$]

(i) For curve $y = 5 - e^{2x}$,

y-intercept $= 5 - e^0 = 4$

x-intercept:

$y = 0$

$5 - e^{2x} = 0$

$e^{2x} = 5$

$2x = \ln 5$

$\Rightarrow x = \dfrac{1}{2}\ln 5$

Let $A(0, 4)$, $B\left(\dfrac{1}{2}\ln 5, 0\right)$

∴ equation of AB:

$$y - 4 = \left(\frac{4 - 0}{0 - \frac{1}{2}\ln 5}\right)(x - 0)$$

$$y = -\frac{8}{\ln 5}x + 4$$

At $(\ln 5, k)$,

$$k = -\frac{8}{\ln 5}(\ln 5) + 4 = -4$$

(ii) $x = \ln\sqrt{9 - x}$

$x = \dfrac{1}{2}\ln(9 - x)$

$2x = \ln(9 - x)$

$e^{2x} = 9 - x$

$x - 9 = -e^{2x}$

$x - 9 + 5 = 5 - e^{2x}$

$x - 4 = 5 - e^{2x}$

∴ equation of straight line is $y = x - 4$.

Question 6

[Ans: (i) prove (ii) prove (iii) prove]

(i) $\angle XBA = \angle AXY$ (Tangent-chord th.)
$\therefore \angle XBA = \angle BXA$
$\Rightarrow \Delta ABX$ is an isosceles Δ (base ∠ of isosc. Δ)
$\therefore AX = AB$ (sides of isosc. Δ) (proven)

(ii) $\angle BCD = \angle BXA$ (∠s in the same segment)
$\angle XCD = \angle XBA$ (∠s in the same segment)
$\therefore \angle XCD = \angle BCD$
$\Rightarrow CD$ bisects $\angle XCB$ (proven)

(iii) $\angle XCD = \angle BCA$ (part (ii))
$\angle CXD = \angle BAC$ (∠s in the same segment)
$\therefore \Delta CDX$ and ΔCBA are similar. (proven)

Question 7

[Ans: (i) $y=\frac{1}{3}x+\frac{5}{6}$ (ii) $\frac{99}{8}$ unit²]

(i) When $y = 0$,

$$\sqrt{2x+5} = 0 \Rightarrow x = -\frac{5}{2}$$

$$\therefore Q\left(-\frac{5}{2}, 0\right)$$

$$y = \sqrt{2x+5} = (2x+5)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(2x+5)^{-\frac{1}{2}}(2) = \frac{1}{\sqrt{2x+5}}$$

When $x = 2$,

$$\frac{dy}{dx} = \frac{1}{\sqrt{4+5}} = \frac{1}{3}$$

Equation of QR:

$$(y-0) = \frac{1}{3}\left(x+\frac{5}{2}\right)$$

$$y = \frac{1}{3}x + \frac{5}{6}$$

(ii) When $x = 2$,

$$y = \frac{1}{3}(2) + \frac{5}{6} = \frac{3}{2}$$

$$\therefore R\left(2, \frac{3}{2}\right)$$

$$\Rightarrow T\left(2, -\frac{3}{2}\right)$$

Area of triangle QST

$$= \frac{1}{2}\left(2+\frac{5}{2}\right)\left(\frac{3}{2}\right) = \frac{27}{8}$$

Area below the curve and the x-axis

$$= \int_{-\frac{5}{2}}^{2} (2x+5)^{\frac{1}{2}}\,dx$$

$$= \left[\frac{(2x+5)^{\frac{3}{2}}}{(2)\left(\frac{3}{2}\right)}\right]_{-\frac{5}{2}}^{2}$$

$$= \frac{1}{3}(9)^{\frac{3}{2}} - \frac{1}{3}(0) = 9$$

∴ Area of shaded region

$$= \frac{27}{8} + 9 = \frac{99}{8}$$

Question 8

[Ans: (i) $P: v=\frac{3t}{2}+9$; $Q: v=t+\frac{t^2}{4}$ (ii) $P: s=\frac{3t^2}{4}+9t$; $Q: s=\frac{t^2}{2}+\frac{t^3}{12}$ (iii) 216 m

(iv) P: 27 m/s; Q: 48 m/s]

(i) For particle P:

$a = 1.5$

$v = \int 1.5\,dt = 1.5t + c = \frac{3t}{2} + c$

When $t = 0$,

$v = 9 \Rightarrow c = 9$

$\therefore v = \frac{3t}{2} + 9$

For particle Q:

$a = 1 + \frac{t}{2}$

$v = \int 1 + \frac{t}{2} dt = t + \frac{t^2}{4} + c$

When $t = 0$,

$v = 0 \Rightarrow c = 0$

$\therefore v = t + \frac{t^2}{4}$

(ii) For particle P:

$s = \int \frac{3t}{2} + 9\,dt = \frac{3t^2}{4} + 9t + c$

When $t = 0$,

$s = 0 \Rightarrow c = 0$

$\therefore s = \frac{3t^2}{4} + 9t$

For particle Q:

$s = \int t + \frac{t^2}{4} dt = \frac{t^2}{2} + \frac{t^3}{12} + c$

When $t = 0$,

$s = 0 \Rightarrow c = 0$

$\therefore s = \frac{t^2}{2} + \frac{t^3}{12}$

(iii) When Q collides with P,

$\frac{3t^2}{4} + 9t = \frac{t^2}{2} + \frac{t^3}{12}$

$\frac{t^3}{12} - \frac{t^2}{4} - 9t = 0$

$\frac{t}{12}(t^2 - 3t - 108) = 0$

$\frac{t}{12}(t - 12)(t + 9) = 0$

$t = -9$ (NA) or 0 or 12

For particle P, when $t = 12$,

$s = \frac{3(12)^2}{4} + 9(12) = 216$

$\therefore Q$ collides with P at a distance of 216 m from O.

(iv) When $t = 12$,

$P: v = \frac{3(12)}{2} + 9 = 27$

$Q: v = (12) + \frac{(12)^2}{4} = 48$

P and Q will be travelling at 27 m/s and 48 m/s respectively.

Question 9

[Ans: (i) 20 (ii) $y = -2x + 30$; $D(15, 0)$ (iii) show]

(i) $AB = BC$

$\sqrt{(8-0)^2 + (14-5)^2} = \sqrt{(8-k)^2 + (14-15)^2}$

$145 = 64 - 16k + k^2 + 1$

$k^2 - 16k - 80 = 0$

$(k+4)(k-20) = 0$

$k = -4$ (NA) or $k = 20$

(ii) Since $AB = BC$ and $AD = CD$, $\therefore BD \perp AC$

Gradient of AC

$= \frac{15-5}{20-0} = \frac{1}{2}$

Equation of BD:

$y - 14 = -2(x - 8)$

$y = -2x + 30$

Let $y = 0$,

$-2x + 30 = 0$

$x = 15$

$\therefore D(15, 0)$

(iii) Area of triangle ABC

$= \frac{1}{2}\begin{vmatrix} 0 & 20 & 8 & 0 \\ 5 & 15 & 14 & 5 \end{vmatrix}$

$= \frac{1}{2}[(320) - (220)]$

$= 50$

Area of quadrilateral $ABCD$

$= \frac{1}{2}\begin{vmatrix} 0 & 15 & 20 & 8 & 0 \\ 5 & 0 & 15 & 14 & 5 \end{vmatrix}$

$= \frac{1}{2}[(545) - (195)]$

$= 175$

$\frac{\text{Area of ABC}}{\text{Area of ABCD}} = \frac{50}{175} = \frac{2}{7}$ (shown)

Question 10

[Ans: (i) $A = -5$, $B = 4$; show (ii) $\frac{2x}{x^2+4}$ (iii) $-\frac{5}{2}\ln|2x+1| + 2\ln(x^2+4) + C$]

(i) Given $\frac{3x^2+4x-20}{(2x+1)(x^2+4)} = \frac{A}{2x+1} + \frac{Bx+C}{x^2+4}$

$3x^2 + 4x - 20 = A(x^2+4) + (Bx+C)(2x+1)$

When $x = -\frac{1}{2}$,

$$3\left(-\frac{1}{2}\right)^2 + 4\left(-\frac{1}{2}\right) - 20 = A\left[\left(-\frac{1}{2}\right)^2 + 4\right]$$

$A = -5$

When $x = 0$,

$-20 = -5(4) + C \Rightarrow C = 0$ (shown)

When $x = 1$,

$3 + 4 - 20 = -5(5) + B(3)$

$B = 4$

(ii) $\frac{d}{dx}\left[\ln(x^2+4)\right]$

$= \frac{1}{x^2+4}(2x) = \frac{2x}{x^2+4}$

(iii) $\int \frac{3x^2+4x-20}{(2x+1)(x^2+4)} dx$

$= \int -\frac{5}{2x+1} + \frac{4x}{x^2+4}\, dx$

$= -5\int \frac{1}{2x+1}\, dx + 2\int \frac{2x}{x^2+4}\, dx$

$= -\frac{5}{2}\ln|2x+1| + 2\ln(x^2+4) + C$

Question 11

[Ans: (i) $A\left(\frac{\pi}{2},5\right)$, $B\left(\frac{3\pi}{2},-1\right)$, $C(\pi,-4)$ (ii) $\alpha = 0.644$, $k=\frac{2}{5}$ (iii) 0.516, 4.48]

(i) By observation,

$A\left(\frac{\pi}{2},5\right), B\left(\frac{3\pi}{2},-1\right), C(\pi,-4)$

(ii) Let $4\cos x - 3\sin x = R\cos(x+\alpha)$

$= R\cos x\cos\alpha - R\sin x\sin\alpha$

$R\cos\alpha = 4 \quad (1)$

$R\sin\alpha = 3 \quad (2)$

$(2)/(1): \tan\alpha = \frac{3}{4} \Rightarrow \alpha = 0.64350$

$(1)^2 + (2)^2: R^2 = 4^2 + 3^2 \Rightarrow R = 5$

$\therefore 4\cos x - 3\sin x = 5\cos(x + 0.64350)$

$4\cos x = 2 + 3\sin x$

$4\cos x - 3\sin x = 2$

$5\cos(x + 0.64350) = 2$

$\cos(x + 0.64350) = \frac{2}{5}$

$\therefore \alpha = 0.644, k = \frac{2}{5}$

(iii) $\cos(x + 0.64350) = \frac{2}{5}$

Basic ∠

$= \cos^{-1}\frac{2}{5} = 1.1593$

$0 \le x \le 2\pi \Rightarrow 0.64350 \le x + 0.64350 \le 6.9267$

$x + 0.64350 = 1.1593, 2\pi - 1.1593$

$x + 0.64350 = 1.1593, 5.1239$

$x = 0.51578, 4.4804$

$\therefore$ x-coordinate of D and E are 0.516 and 4.48 respectively.