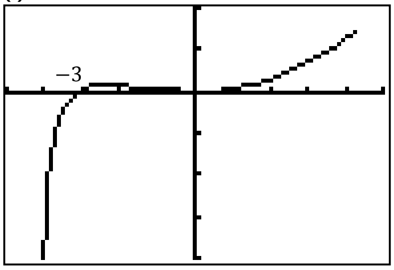
CAMBRIDGE/2009/PAPER2

1. (i) From GC,



(ii) $\frac{dy}{dt} = 3t^2 + 2t$; $\frac{dx}{dt} = 2t + 4$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 + 2t}{2t + 4}$

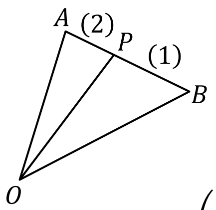
When $t = 2$,
$x = 2^2 + 4(2) = 12$
$y = 2^3 + 2^2 = 12$
$\frac{dy}{dx} = \frac{3(2)^2 + 2(2)}{2(2) + 4} = 2$
$\therefore l: y - 12 = 2(x - 12)$
$y = 2x - 12$

(iii) $x = t^2 + 4t \quad (1)$
$y = t^3 + t^2 \quad (2)$
$y = 2x - 12 \quad (3)$

Sub. (1) & (2) into (3)
$t^3 + t^2 = 2(t^2 + 4t) - 12$
$t^3 - t^2 - 8t + 12 = 0$
$t = 2$ or $t = -3$

When $t = -3$,
$x = (-3)^2 + 4(-3) = -3$
$y = (-3)^3 + (-3)^2 = -18$
$\therefore Q(-3, -18)$

2. (i)



$\overrightarrow{OP} = \frac{\overrightarrow{OA} + 2\overrightarrow{OB}}{1 + 2} = \frac{\begin{pmatrix}14\\14\\14\end{pmatrix} + 2\begin{pmatrix}11\\-13\\2\end{pmatrix}}{3}$
$= \begin{pmatrix}12\\-4\\6\end{pmatrix}$
$\therefore P(12, -4, 6)$

(ii) $\overrightarrow{AB} = \begin{pmatrix}11\\-13\\2\end{pmatrix} - \begin{pmatrix}14\\14\\14\end{pmatrix} = \begin{pmatrix}-3\\-27\\-12\end{pmatrix}$

$\overrightarrow{AB} \cdot \overrightarrow{OP} = \begin{pmatrix}-3\\-27\\-12\end{pmatrix} \cdot \begin{pmatrix}12\\-4\\6\end{pmatrix}$
$= -36 + 108 - 72 = 0$
$\therefore \overrightarrow{AB} \perp \overrightarrow{OP}$ (shown)

(iii)
$\boldsymbol{c} = \frac{\begin{pmatrix}12\\-4\\6\end{pmatrix}}{\left|\begin{pmatrix}12\\-4\\6\end{pmatrix}\right|} = \frac{1}{14}\begin{pmatrix}12\\-4\\6\end{pmatrix} = \frac{1}{7}\begin{pmatrix}6\\-2\\3\end{pmatrix}$

$|\boldsymbol{a} \cdot \boldsymbol{c}|$ is the length of projection of $\overrightarrow{OA}$ onto $\overrightarrow{OP}$.

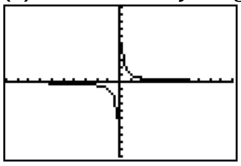
(iv) $\boldsymbol{a} \times \boldsymbol{p} = \begin{pmatrix}14\\14\\14\end{pmatrix} \times \begin{pmatrix}12\\-4\\6\end{pmatrix} = \begin{pmatrix}140\\84\\-224\end{pmatrix}$
$|\boldsymbol{a} \times \boldsymbol{p}|$ is the area of parallelogram formed by vectors $\boldsymbol{a}$ and $\boldsymbol{p}$.

Area of ΔOAP
$= \frac{1}{2}|\boldsymbol{a} \times \boldsymbol{p}|$
$= \frac{1}{2}\left|\begin{pmatrix}140\\84\\-224\end{pmatrix}\right| = \frac{1}{2}\sqrt{76832} = 98\sqrt{2}$

3. (i) Let $y = \frac{ax}{bx - a}$
$bxy - ay = ax$
$bxy - ax = ay$
$x = \frac{ay}{by - a}$
$\therefore f^{-1}(x) = \frac{ax}{bx - a}, x \in \mathbb{R}, x \neq \frac{a}{b}$

$f^2(x) = ff(x) = ff^{-1}(x) = x$
$R_{f^2} = \left(-\infty, \frac{a}{b}\right) \cup \left(\frac{a}{b}, \infty\right)$

(ii) From GC, $y = g(x)$



$R_g = (-\infty, 0) \cup (0, \infty)$

$D_f = \left(-\infty, \frac{a}{b}\right) \cup \left(\frac{a}{b}, \infty\right), \frac{a}{b} \neq 0$

$\therefore R_g \not\subseteq D_f \Rightarrow$ function fg does not exist.

(iii) $f^{-1}(x) = x$

$\frac{ax}{bx - a} = x$

$bx^2 - 2ax = 0$

$x(bx - 2a) = 0$

$x = 0$ or $x = \frac{2a}{b}$

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4. (i) $\frac{d^2n}{dt^2} = 10 - 6t$

$\frac{dn}{dt} = \int 10 - 6t \, dt = 10t - 3t^2 + A$

$n = \int 10t - 3t^2 + A \, dt$

$= 5t^2 - t^3 + At + B$

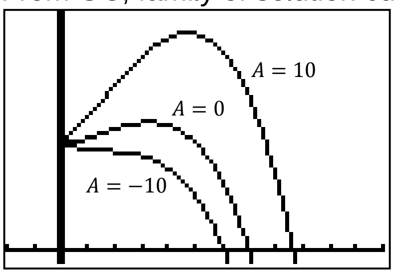
When $t = 0$,

$n = 100$

$\Rightarrow B = 100$

$\therefore 5t^2 - t^3 + At + 100$

From GC, family of solution curves:



(ii) $\frac{dn}{dt} = 3 - 0.02n$

$\frac{1}{3 - 0.02n} \frac{dn}{dt} = 1$

$\int \frac{1}{3 - 0.02n} dn = \int dt$

$-\frac{1}{0.02} \ln|3 - 0.02n| = t + A$

$\ln|3 - 0.02n| = -\frac{t}{50} + B$

$3 - 0.02n = e^{-\frac{t}{50} + B} = Ce^{-\frac{t}{50}}$

$n = 50\left(3 - Ce^{-\frac{t}{50}}\right)$

When $t = 0$,

$n = 100$

$50(3 - C) = 100 \Rightarrow C = 1$

$\therefore n = 50\left(3 - e^{-\frac{t}{50}}\right)$

When $n \to \infty$, $n \to 50(3) = 150$

$\therefore$ The population will eventually become 150000.

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5. The manager can employ surveyors to stand at the entrance to the cinema and survey the first 100 cinema-goers that enter the cinema.
One disadvantage of quota sampling is that it is biased in nature as the surveyor may interview cinema-goers who are more willing or easier to reach.

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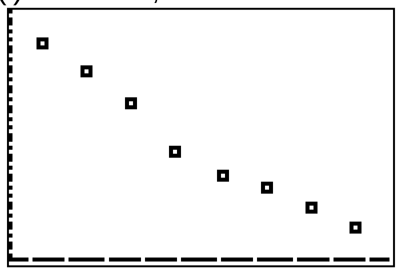
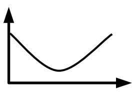
6. (i) From GC,



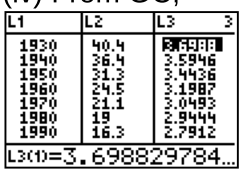
(ii) From the scatter diagram, a linear model is Year 1930 to Year 2000. However it should be noted that linearity should only be limited to this period as further progression of such a linear line will cause "time" to become negative.

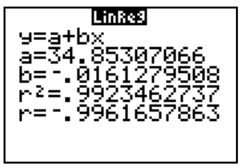
(iii)



A quadratic model is inappropriate as it will imply the time taken to go to infinity in long run.

(iv) From GC,





$\ln t = 34.8531 - 0.01613x$

When $x = 2010$,

$\ln t = 34.8531 - 0.01613(2010)$

$t = e^{2.4317} = 11.4\text{s}$

$\therefore$ The world record time will be 3 min 41.4 s.

The prediction is unreliable because Year 2010 is outside the data range.

7. (i) Required probability
$= (0.25)(0.05) + (0.75)(0.03) = 0.035$

(ii) $f(p)$
$= P(supplied\ by\ A | component\ faulty)$
$= \frac{P(supplied\ by\ A \cap component\ faulty)}{P(component\ faulty)}$
$= \frac{\left(\frac{p}{100}\right)(0.05)}{\left(\frac{p}{100}\right)(0.05) + \left(\frac{100-p}{100}\right)(0.03)}$
$= \frac{0.05p}{0.05p + 3 - 0.03p} = \frac{0.05p}{0.02p + 3}$
(shown)

$f'(p)$
$= \frac{(0.02p+3)(0.05) - (0.05p)(0.02)}{(0.02p+3)^2}$
$= \frac{0.15}{(0.02p+3)^2} > 0$
$\therefore$ For $0 \le p \le 100$, $f(p)$ is an increasing function.

An increasing $f(p)$ means that if the percentage of the components bought from supplier A increases, the probability that a faulty component is supplied by A will also increase.

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8. (i) No. of ways $= \frac{8!}{3!} = 6720$

(ii) _E_L_E_V_A_E_
No. of ways
$= \frac{6!}{3!}\ {}^7C_2 2! = 5040$

(iii) Case I: CVCVCVCV
No. of ways
$= \frac{4!\,4!}{3!} = 96$

Case II: VCVCVCVC
No. of ways
$= \frac{4!\,4!}{3!} = 96$

Total no. of ways $= 96 + 96 = 192$

(iv) Case I: E_ _E_ _E_
No. of ways
$= 5! = 120$

Case II: _E_ _E_ _E
No. of ways
$= 5! = 120$

Case III: E_ _ _E_ _E
No. of ways
$= 5! = 120$

Case IV: E_ _ E_ _ _E
No. of ways
$= 5! = 120$

Total no. of ways $= 120 \times 4 = 480$

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9. $\overline{M} \sim N\left(2.5, \frac{0.1^2}{n}\right)$

$P(\overline{M} > 2.53) = 0.0668$
$P(\overline{M} < 2.53) = 0.9332$
$P\left(Z < \frac{2.53 - 2.5}{0.1/\sqrt{n}}\right) = 0.9332$
$P(Z < 0.3\sqrt{n}) = 0.9332$
From GC,
$0.3\sqrt{n} = 1.5001$
$n = 25$

(ii) Let S be the thickness of a statistics textbook.
$S \sim N(2.0, 0.08^2)$

Let $X = M_1 + M_2 + \cdots + M_{21}$
$E(X) = E(M_1 + M_2 + \cdots + M_{21})$
$= 21E(M) = 21(2.5) = 52.5$
$Var(X) = Var(M_1 + M_2 + \cdots + M_{21})$
$= 21Var(M) = 21(0.1^2) = 0.21$

Let $Y = S_1 + S_2 + \cdots + S_{24}$
$E(Y) = E(S_1 + S_2 + \cdots + S_{24})$
$= 24E(S) = 24(2) = 48$
$Var(Y) = Var(S_1 + S_2 + \cdots + S_{24})$
$= 24Var(S)$
$= 24(0.81^2) = 0.1536$

$E(X+Y) = E(X) + E(Y)$
$= 52.5 + 48 = 100.5$
$Var(X+Y) = Var(X) + Var(Y)$
$= 0.21 + 0.1536 = 0.3636$

$\therefore X + Y \sim N(100.5, 0.3636)$
Required probability
$= P(X + Y < 100) = 0.203$

(iii) *[continue on next page]*

(iii) Let $A = S_1 + S_2 + S_3 + S_4$
$E(A) = 4E(S) = 4(2) = 8$
$Var(A) = 4Var(S) = 4(0.08^2) = 0.0256$

$E(A - 3M) = E(A) - 3E(M)$
$= 8 - 3(2.5) = 0.5$
$Var(A - 3M) = Var(A) + 3^2 Var(M)$
$= 0.0256 + 3^2(0.1^2)$
$= 0.1156$

$\therefore A - 3M \sim N(0.5, 0.1156)$

Required probability
$= P(A < 3M) = P(A - 3M < 0) = 0.0707$

(iv) It is assumed that the thickness of the mechanics textbooks and the thickness of the statistics textbook are independent of each other.

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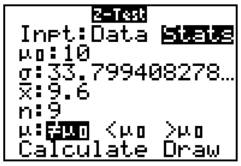
10. (i) Unbiased estimate of population mean
$= \frac{86.4}{9} = 9.6$

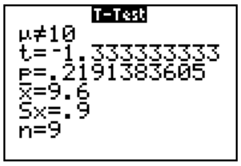
Unbiased estimate of population variance, s^2
$= \frac{1}{8}\left(835.92 - \frac{(86.4)^2}{9}\right) = 0.81$

(ii) It is assumed that the mass of the packets of sugar follows a normal distribution.
Let μ be the mean mass of a packet of sugar.
$H_0: \mu = 10$
$H_1: \mu \neq 10$
$n = 9$
$\bar{x} = 9.6$
$s^2 = 0.81$

Test statistics, $T = \frac{\overline{X} - \mu}{s/\sqrt{n}} \sim t(8)$

From GC,





p-value $= 0.219$
Since p-value > 0.05, there is insufficient evidence to reject H_0 at 5% level of significance. i.e. the mean mass of the packets of sugar may not be incorrect.

Central Limit Theorem does not apply as the sample size is too small.

(iii) A Z-Test will be used instead.

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11. (i) Assumptions:
(a) The probability of each observation remains the same.
(b) Each observation is independent of another.

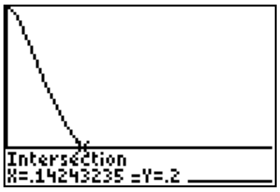
(ii) $R \sim B(20, 0.15)$
$P(4 \leq R < 8) = P(4 \leq R \leq 7)$
$= P(R \leq 7) - P(R \leq 3)$
$= 0.346$

(iii) $R \sim B(240, 0.3)$
$np = 72 > 5$, $n(1-p) = 168 > 5$
$np(1-p) = 50.4$
$\therefore R \sim N(72, 50.4)$ approx.
$P(R < 60) = P(R \leq 59)$
$= P(R < 59.5)\ (c.c.)$
$= 0.0391$

(iv) $R \sim B(240, 0.02)$
$np = 4.8 < 5$
$\therefore R \sim Po(4.8)$ approx.
$P(R = 3) = 0.1571$

A Poisson approximation is appropriate because n is 240 while p is just 0.02, which makes the expected outcome a very small value. Thus it can be considered as a close approximate to an average over a large population.

(v) $R \sim B(20, p)$
$P(R = 0\ or\ 1) = 0.2$
$P(R = 0) + P(R = 1) = 0.2$
$\binom{20}{0}(1-p)^{20}p^0 + \binom{20}{1}(1-p)^{19}p^1$
$= 0.2$
$(1-p)^{20} + 20p(1-p)^{19} = 0.2$
From GC,



$p = 0.142$