CAMBRIDGE/2009/PAPER1

1. Let $f(x) = 2x^3 + ax^2 + bx + 3$

   $f(1) = 0$
   $2 + a + b + 3 = 0$
   $a + b = -5 \quad (1)$

   $f(-2) = 15$
   $2(-2)^3 + a(-2)^2 + b(-2) + 3 = 15$
   $4a - 2b = 28$
   $2a - b = 14 \quad (2)$

   $(1) + (2)$,
   $3a = 9 \Rightarrow a = 3$

   Sub. $a = 3$ into (1),
   $3 + b = -5 \Rightarrow b = -8$

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2. Given $y = \dfrac{\ln x}{x}$

   $$\frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - \ln x\,(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

   Let $\dfrac{dy}{dx} > 0$
   $\dfrac{1 - \ln x}{x^2} > 0$
   $1 - \ln x > 0$
   $\ln x < 1$
   $0 < x < e$

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3. $x\sqrt{24} = x\sqrt{3} + \sqrt{6}$
   $x\sqrt{24} - x\sqrt{3} = \sqrt{6}$

   $$x = \frac{\sqrt{6}}{\sqrt{24} - \sqrt{3}}\left(\frac{\sqrt{24} + \sqrt{3}}{\sqrt{24} + \sqrt{3}}\right)$$
   $$= \frac{\sqrt{6}\sqrt{24} + \sqrt{6}\sqrt{3}}{24 - 3}$$
   $$= \frac{\sqrt{144} + \sqrt{18}}{21}$$
   $$= \frac{12 + 3\sqrt{2}}{21} = \frac{4 + \sqrt{2}}{7}$$

   $\therefore a = 4,\ b = 2$

4. (i) $\lg(x + 14) - \lg(x - 2) = 2\lg 5$

   $\lg\left(\dfrac{x + 14}{x - 2}\right) = \lg 25$
   $\dfrac{x + 14}{x - 2} = 25$
   $x + 14 = 25x - 50$
   $24x = 64 \Rightarrow x = \dfrac{8}{3}$

   (ii) $\log_2 y + \log_4 y = 6$
   $\log_2 y + \dfrac{\log_2 y}{\log_2 4} = 6$
   $\log_2 y + \dfrac{1}{2}\log_2 y = 6$
   $\dfrac{3}{2}\log_2 y = 6$
   $\log_2 y = 4 \Rightarrow y = 2^4 = 16$

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5. When $x = 0$,
   $y = 1 - 3\tan(0) = 1$
   $\dfrac{dy}{dx} = -3\sec^2(0) = -3$
   ∴ Equation of normal:
   $$y - 1 = -\left(\frac{1}{-3}\right)(x - 0)$$
   $$y = \frac{1}{3}x + 1$$

   At $(k, 3)$,
   $3 = \dfrac{1}{3}k + 1 \Rightarrow k = 6$

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6. (i) When $c = -20$,
   $y \le 0$
   $2x^2 - 6x - 20 \le 0$
   $x^2 - 3x - 10 \le 0$
   $(x - 5)(x + 2) \le 0$
   $-2 \le x \le 5$

   (ii) $y + 2x = 8 \Rightarrow y = -2x + 8$
   Let $-2x + 8 = 2x^2 - 6x + c$
   $2x^2 - 4x + c - 8 = 0$
   Discriminant $= 0$
   $(-4)^2 - 4(2)(c - 8) = 0$
   $8(c - 8) = 16$
   $c = 10$

7. $x^2 + 2y^2 + 5x = 68 \quad (1)$
$2y + 3x = 9$
$\Rightarrow y = \dfrac{9-3x}{2} \quad (2)$

Sub. (2) into (1),
$x^2 + 2\left(\dfrac{9-3x}{2}\right)^2 + 5x = 68$
$x^2 + \dfrac{1}{2}(81 - 54x + 9x^2) + 5x - 68 = 0$
$2x^2 + 81 - 54x + 9x^2 + 10x - 136 = 0$
$11x^2 - 44x - 55 = 0$
$x^2 - 4x - 5 = 0$
$(x+1)(x-5) = 0$
$x = -1$ or $x = 5$

When $x = -1$,
$y = \dfrac{9-3(-1)}{2} = 6$
When $x = 5$,
$y = \dfrac{9-3(5)}{2} = -3$
Coordinates of intersections are $(-1, 6)$ and $(5, -3)$.
∴ Midpoint
$= \left(\dfrac{-1+5}{2}, \dfrac{6-3}{2}\right) = \left(2, \dfrac{3}{2}\right)$

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8. (i) LHS
$= -2\sin\left(\dfrac{3x+x}{2}\right)\sin\left(\dfrac{3x-x}{2}\right)$
$= -2\sin 2x \sin x$
$= -2(2\sin x\cos x)\sin x$
$= -4\sin^2 x\cos x = RHS$

(ii) $\cos 3x + 2\cos x = 0$
$\cos x - 4\sin^2 x\cos x + 2\cos x = 0$
$3\cos x - 4\sin^2 x\cos x = 0$
$\cos x\,(3 - 4\sin^2 x) = 0$
$\cos x = 0$ or $\sin x = \pm\dfrac{\sqrt{3}}{2}$
$x = \dfrac{\pi}{2}$ $\qquad x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

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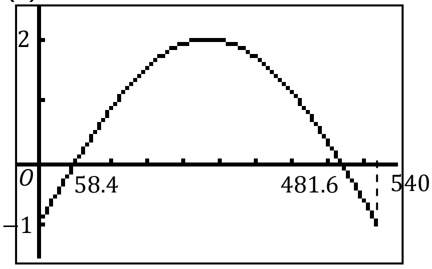
9. (i) Maximum $f(x) = 2$
Minimum $f(x) = -4$

(ii) Amplitude = 3

(iii) Period $\dfrac{360}{1/3} = 1080°$

(iv) $f(x) = 0$
$3\sin\left(\dfrac{x}{3}\right) - 1 = 0$
$\sin\left(\dfrac{x}{3}\right) = \dfrac{1}{3} \Rightarrow x = 3\sin^{-1}\dfrac{1}{3} = 58.4°$

(v)



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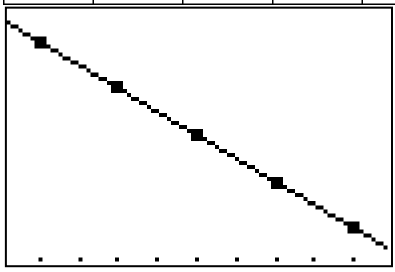
10. (i)

t	2	4	6	8	10
$\ln m$	3.88	3.73	3.58	3.42	3.28



(ii) $m = m_0 e^{-kt}$
$\ln m = \ln m_0 - kt$

From graph,
Gradient = -0.0750
$\Rightarrow -k = -0.0750$
$\Rightarrow k = 0.0750$

$\ln m_0 = 4.03 \Rightarrow m_0 = 56.3$

(iii) Initially,
$m = m_0 = 56.3$
When mass is halved,
$m = \dfrac{56.3}{2} = 28.1$
$\ln m = \ln 28.1 = 3.34$
From graph, when $\ln m = 3.34$,
$t = 9.20$.
∴ It takes approximately 9.20 hours for the mass of the substance to be halved.

11. (i) Gradient of AD

$= \frac{6+2}{0-2} = -4$

$\therefore$ Equation of AB:

$y - 6 = -\frac{1}{(-4)}(x - 0)$

$y = \frac{1}{4}x + 6 \quad (1)$

(ii) $y = x \quad (2)$

Sub. (2) into (1),

$x = \frac{1}{4}x + 6 \Rightarrow \frac{3}{4}x = 6 \Rightarrow x = 8$

$y = 8$

$\therefore B(8, 8)$

(iii) By similar triangles,

x-coordinate of $C = 2 + 2(8) = 18$

y-coordinate of $C = -2 + 2(2) = 2$

$\therefore C(18, 2)$

(iv) Area of trapezium $ABCD$

$= \frac{1}{2}\begin{vmatrix} 0 & 2 & 18 & 8 & 0 \\ 6 & -2 & 2 & 8 & 6 \end{vmatrix}$

$= \frac{1}{2}[(0 + 4 + 144 + 48) - (12 - 36 + 16 + 0)]$

$= \frac{1}{2}(196 - (-8)]$

$= 102$ units^2

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12. (i) $y = (2x - 1)(4x + 1)^{\frac{1}{2}}$

$\frac{dy}{dx} = (2x - 1)\left(\frac{1}{2}\right)(4x + 1)^{-\frac{1}{2}}(4) + (2)(4x + 1)^{\frac{1}{2}}$

$= \frac{2(2x - 1)}{\sqrt{4x + 1}} + 2\sqrt{4x + 1}$

$= \frac{4x - 2 + 2(4x + 1)}{\sqrt{4x + 1}}$

$= \frac{12x}{\sqrt{4x + 1}}$

(ii) When $x = 2$,

$\frac{dy}{dx} = 2$

$\frac{dy}{dx} = \frac{12(2)}{\sqrt{4(2) + 1}} = 8$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = \frac{dy/dt}{dy/dx} = \frac{2}{8} = \frac{1}{4}$

(iii) From part (i),

$\int_0^2 \frac{12x}{\sqrt{4x + 1}}\,dx = \left[(2x - 1)\sqrt{4x + 1}\right]_0^2$

$4\int_0^2 \frac{3x}{\sqrt{4x + 1}}\,dx = (3)\sqrt{9} - (-1)\sqrt{1}$

$\int_0^2 \frac{3x}{\sqrt{4x + 1}}\,dx = \frac{1}{4}(10) = \frac{5}{2}$