CAMBRIDGE/2008/PAPER2

1. (i) When he first bought it, $t = 0$,
$V = 10000e^0 = 10000$

(ii) When $t = 12$,
$V = 4000$
$10000e^{-12p} = 4000$
$e^{-12p} = \frac{2}{5}$
$-12p = \ln\frac{2}{5} \Rightarrow p = -\frac{1}{12}\ln\frac{2}{5}$
$\therefore V = 10000e^{\frac{t}{12}\ln\frac{2}{5}}$
When $t = 18$,
$V = 10000e^{\frac{18}{12}\ln\frac{2}{5}} = 2530$

(iii) Let $V = 1000$
$10000e^{\frac{t}{12}\ln\frac{2}{5}} = 1000$
$e^{\frac{t}{12}\ln\frac{2}{5}} = \frac{1}{10}$
$\frac{t}{12}\ln\frac{2}{5} = \ln\frac{1}{10} \Rightarrow t = 30$

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2. $2x^2 - 4x + 3 = 0$
$\alpha + \beta = -\left(\frac{-4}{2}\right) = 2$
$\alpha\beta = \frac{3}{2}$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
$= (2)^2 - 2\left(\frac{3}{2}\right) = 1$

Sum of new roots
$= (\alpha^2 + 2) + (\beta^2 + 2)$
$= (\alpha^2 + \beta^2) + 4$
$= (1) + 4 = 5$

Product of new roots
$= (\alpha^2 + 2)(\beta^2 + 2)$
$= \alpha^2\beta^2 + 2\alpha^2 + 2\beta^2 + 4$
$= (\alpha\beta)^2 + 2(\alpha^2 + \beta^2) + 4$
$= \left(\frac{3}{2}\right)^2 + 2(1) + 4 = \frac{33}{4}$

New equation:
$x^2 - 5x + \frac{33}{4} = 0$
$4x^2 - 20x + 33 = 0$

3. (i) LHS
$= \tan A + \cot A$
$= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$
$= \frac{\sin^2 A + \cos^2 A}{\sin A \cos A}$
$= \frac{1}{\frac{1}{2}\sin 2A}$
$= 2\operatorname{cosec} 2A =$ RHS

(ii) $\tan A + \cot A = 3$
$2\operatorname{cosec} 2A = 3$
$\sin 2A = \frac{2}{3}$

Basic ∠
$= \sin^{-1}\frac{2}{3} = 41.810°$

$0° < 2A < 720°$

$\therefore 2A = 41.810°, 180° - 41.810°, 41.810° + 360°, (180° - 41.810°) + 360°$
$A = 20.9°, 69.1°, 200.9°, 249.1°$

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4. (i) $2 + \log_3(3x - 7) = \log_3(2x - 3)$
$\log_3(2x - 3) - \log_3(3x - 7) = 2$
$\log_3\left(\frac{2x - 3}{3x - 7}\right) = 2$
$\frac{2x - 3}{3x - 7} = 3^2 = 9$
$2x - 3 = 27x - 63$
$25x = 60 \Rightarrow x = \frac{12}{5}$

(ii) $3\log_5 y - \log_y 5 = 2$
$3\log_5 y - \frac{\log_5 5}{\log_5 y} = 2$
$3\log_5 y - \frac{1}{\log_5 y} = 2$
Let $u = \log_5 y$
$3u - \frac{1}{u} = 2$
$3u^2 - 2u - 1 = 0$
$(3u + 1)(u - 1) = 0$
$u = -\frac{1}{3}$ or $u = 1$
$\log_5 y = -\frac{1}{3}$
$y = 5^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{5}}$

$u = 1$
$\log_5 y = 1$
$y = 5^1 = 5$

5. (i) $f(x)$
$= 2(x+1)(x-2)(x^2-3x+1)$
$= 2(x^2-x-2)(x^2-3x+1)$
$= 2(x^4-3x^3+x^2-x^3+3x^2-x-2x^2+6x-2)$
$= 2(x^4-4x^3+2x^2+5x-2)$
$= 2x^4-8x^3+4x^2+10x-4$

(ii) For x^2-3x+1,
Discriminant
$= (-3)^2-4(1)(1) = 5 > 0$
⇒ Quadratic factor will contribute to 2 additional roots of $f(x)$.
∴ $f(x)$ has 4 real roots.

(iii) Remainder
$= f\left(\frac{1}{2}\right)$
$= 2\left(\frac{1}{2}\right)^4 - 8\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^2 + 10\left(\frac{1}{2}\right) - 4$
$= \frac{9}{8}$

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6. (i) $OA = OC$ (radii of circle)
$AE = CE$ (tangents from ext. point)
OE is a common side of Δs
∴ ΔAEO is congruent to ΔCEO (SSS congruence)

(ii) $\angle AOE = \frac{1}{2}\angle AOC$ (tangents from ext. point)
$\angle ABC = \frac{1}{2}\angle AOC$ (∠ at centre = 2∠ at circumference)
∴ $\angle AOE = \angle ABC$
$\angle EAB = \angle DAB$ (common ∠s)

∴ ΔBAD is similar to ΔOAE.
$\frac{AE}{AD} = \frac{AO}{AB} = \frac{1}{2}$ (corr. sides of similar Δs)
∴ $AE = \frac{1}{2}AD$
⇒ E is the mid-point of AD.

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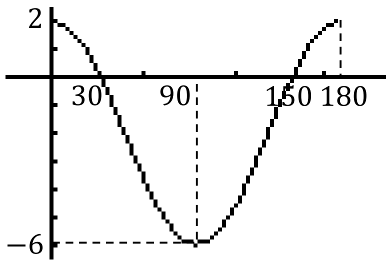
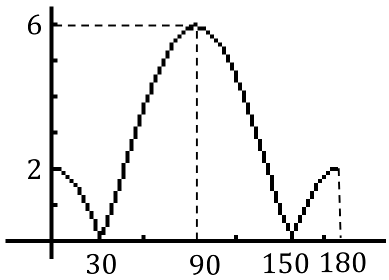
7. (i) Amplitude $= 4$
(ii) Period $= \frac{360°}{2} = 180°$
(iii) Minimum point
$= \left(\frac{180}{2}, -6\right) = (90, -6)$

(iv) Let $y = 0$
$4\cos 2x - 2 = 0$
$\cos 2x = \frac{1}{2}$
Basic ∠
$= \cos^{-1}\frac{1}{2} = 60°$
$0° < 2x < 360°$
$2x = 60°, 360° - 60°$
$x = 30°, 150°$

(v) $y = 4\cos 2x - 2$



(vi) $y = |4\cos 2x - 2|$



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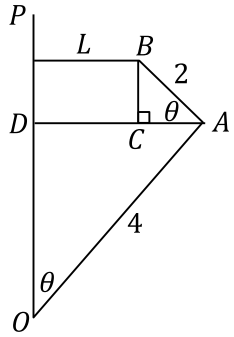
8. (i) Given $y = x^3 - ax + b$
$\frac{dy}{dx} = 3x^2 - a$
At $(2, 0)$,
$\frac{dy}{dx} = 0$
$3(2)^2 - a = 0 \Rightarrow a = 12$

$2^3 - 12(2) + b = 0 \Rightarrow b = 16$

(ii) Let $\frac{dy}{dx} = 0$
$3x^2 - 12 = 0$
$x^2 = 4 \Rightarrow x = \pm 2$
When $x = -2$,
$y = (-2)^3 - 12(-2) + 16 = 32$
∴ Maximum point is at $(-2, 21)$.

(iii) Area $\int_0^2 x^3 - 12x + 16dx$
$= \left[\frac{1}{4}x^4 - 6x^2 + 16x\right]_0^2 = 12$

9. (i)



$\sin\theta = \frac{AD}{4} \Rightarrow AD = 4\sin\theta$

$\cos\theta = \frac{AC}{2} \Rightarrow AC = 2\cos\theta$

$\therefore L = AD - AC = 4\sin\theta - 2\cos\theta$ (shown)

(ii) Let $L = R\sin(\theta - \alpha)$
$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

$R\cos\alpha = 4 \quad (1)$
$R\sin\alpha = 2 \quad (2)$

$(2)/(1)$ $\tan\alpha = \frac{2}{4} \Rightarrow \alpha = 26.565°$
$(1)^2 + (2)^2$ $R^2 = 4^2 + 2^2$
$\Rightarrow R = \sqrt{20} = 2\sqrt{5}$

$\therefore L = 2\sqrt{5}\sin(\theta - 26.6°)$

(iii) $L = 3$
$2\sqrt{5}\sin(\theta - 26.6°) = 3$
$\sin(\theta - 26.6°) = \frac{3}{2\sqrt{5}}$
$\theta = \sin^{-1}\frac{3}{2\sqrt{5}} + 26.6° = 68.7°$

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10. (i) At $P(2,9)$,
$\frac{dy}{dx} = \frac{6}{[2(2)-1]^2} = \frac{2}{3}$
Equation of normal:
$y - 9 = -\frac{3}{2}(x-2)$
$y = -\frac{3}{2}x + 12$
$Q(0,12)$
Let $y = 0$,
$-\frac{3}{2}x + 12 = 0 \Rightarrow x = 8$
$R(8,0)$
Mid-point of QR
$= \left(\frac{0+8}{2}, \frac{12+0}{2}\right) = (4,6)$

(ii) $y = \int \frac{6}{(2x-1)^2}dx = 6\int (2x-1)^{-2}dx$
$y = 6\left[\frac{(2x-1)^{-1}}{(2)(-1)}\right] + c$
$y = -\frac{3}{2x-1} + c = \frac{3}{1-2x} + c$
At $P(2,9)$,
$\frac{3}{1-2(2)} + c = 9 \Rightarrow c = 10$
$\therefore$ Equation of curve:
$y = \frac{3}{1-2x} + 10$

(iii) At $P(2,9)$,
$\frac{dx}{dt} = 0.03$ (given)
$\frac{dy}{dx} = \frac{2}{3}$
$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \left(\frac{2}{3}\right)(0.03) = 0.02$

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11. (i) $OP = \sqrt{8^2 + 6^2} = 10$
Equation of C_1:
$x^2 + y^2 = 10^2$
$x^2 + y^2 = 100 \quad (1)$

(ii) $Q\left(\frac{8}{2}, \frac{-6}{2}\right) \Rightarrow Q(4,-3)$
Equation of C_2:
$(x-4)^2 + (y+3)^2 = 5^2$
$(x-4)^2 + (y+3)^2 = 25$

(iii) Gadient of $OP = \frac{0+6}{0-8} = -\frac{3}{4}$
Equation of AB:
$y + 3 = \frac{4}{3}(x-4)$
$\Rightarrow y = \frac{4}{3}x - \frac{25}{3} \quad (2)$
Sub. (2) into (1)
$x^2 + \left(\frac{4}{3}x - \frac{25}{3}\right)^2 = 100$
$x^2 + \frac{16}{9}x^2 - \frac{200}{9}x + \frac{625}{9} = 100$
$\frac{25}{9}x^2 - \frac{200}{9}x - \frac{275}{9} = 0$
$x^2 - 8x - 11 = 0$
$x = \frac{8 \pm \sqrt{(-8)^2 - 4(1)(-11)}}{2(1)}$
$= \frac{8 \pm \sqrt{108}}{2} = 4 \pm 3\sqrt{3}$
$\therefore$ x-coordinates of A and B are $4 + 3\sqrt{3}$ and $4 - 3\sqrt{3}$ (shown)
$a = 4$, $b = 3$