CAMBRIDGE/2007/PAPER1

1. $\frac{2x^2-x-19}{x^2+3x+2}-1$

$=\frac{2x^2-x-19-(x^2-3x+2)}{x^2+3x+2}$

$=\frac{x^2-4x-21}{x^2+3x+2}$

(proven)

$\frac{2x^2-x-19}{x^2+3x+2}>1, \quad x\neq -2, x\neq -1$

$\frac{2x^2-x-19}{x^2+3x+2}-1>0$

$\frac{x^2-4x-21}{x^2+3x+2}>0$

$\frac{(x-7)(x+3)}{(x+2)(x+1)}>0$

$+ \mid - \mid + \mid - \mid +$

$-3 \quad -2 \quad -1 \quad 7$

$x<-3$ or $-2<x<-1$ or $x>7$

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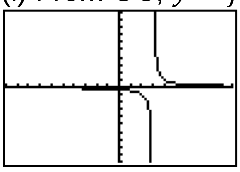
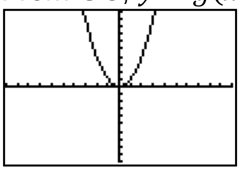
2. (i) From GC, $y=f(x)$:



$R_f=(-\infty,0)\cup(0,\infty)$

From GC, $y=g(x)$:



$R_g=[0,\infty)$

Since $R_g \nsubseteq D_f \Rightarrow fg$ does not exist.

Since $R_f \subseteq D_g \Rightarrow gf$ exists.

$gf: x \mapsto \frac{1}{(x-3)^2}, x\in\mathbb{R}, x\neq 3$

(ii) Let $y=\frac{1}{x-3}$ for $x\neq 3$

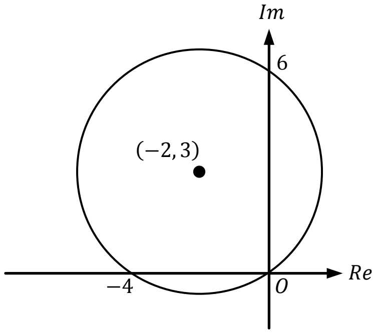
$x-3=\frac{1}{y}$

$x=\frac{1}{y}+3=\frac{1+3y}{y}$

$\therefore f^{-1}: x\mapsto \frac{1+3x}{x}, x\in\mathbb{R}, x\neq 0$

3. (a) $|z+2-3i|=\sqrt{13}$

$\Rightarrow |z-(-2+3i)|=\sqrt{13}$



(b) Let $w=a+ib, w^*=a-ib$

$ww^*+2w=3+4i$

$(a+ib)(a-ib)+2(a+ib)=3+4i$

$a^2+b^2+2a+2ib=3+4i$

$2b=4\Rightarrow b=2$

$a^2+(2)^2+2a=3$

$a^2+2a+1=0$

$(a+1)^2=0\Rightarrow a=-1$

$\therefore w=-1+2i$

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4. $4\frac{dI}{dt}=2-3I$

$\frac{4}{2-3I}\frac{dI}{dt}=1$

$4\int\frac{1}{2-3I}\,dI=\int dt$

$-\frac{4}{3}\ln|2-3I|=t+A$

$\ln|2-3I|=-\frac{3}{4}t+B$

$2-3I=e^{-\frac{3}{4}t+B}=e^Be^{-\frac{3}{4}t}$

$I=\frac{1}{3}\left(2+Ce^{-\frac{3}{4}t}\right)$

When $t=0$, $I=2$,

$2=\frac{1}{3}(2+C)\Rightarrow C=4$

$\therefore I=\frac{1}{3}\left(2+4e^{-\frac{3}{4}t}\right)$

$=\frac{2}{3}\left(1+2e^{-\frac{3}{4}t}\right)$

As $t\to\infty$,

$I=\lim_{t\to\infty}\frac{2}{3}\left(1+2e^{-\frac{3}{4}t}\right)=\frac{2}{3}$

5.

$$\begin{array}{r} 2 \\ x+2\overline{)2x+7} \\ \underline{-(2x+4)} \\ 3 \end{array}$$

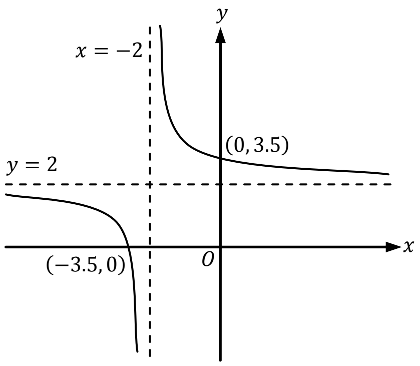
$$\therefore y = 2 + \frac{3}{x+2} \Rightarrow A = 2, B = 3$$

Transformations:

(1) Translate 2 units in the negative x-direction.

(2) Stretch by a factor of 3 parallel to the y-axis.

(3) Translate 2 units in the positive y-direction.

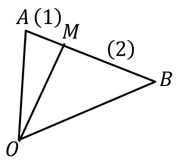


6. (i) $\overrightarrow{OA} = \begin{pmatrix}1\\-1\\2\end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix}2\\4\\1\end{pmatrix}$

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = \begin{pmatrix}1\\-1\\2\end{pmatrix} \cdot \begin{pmatrix}2\\4\\1\end{pmatrix} = 2 - 4 + 2 = 0$$

$\therefore OA \perp OB$

(ii)



$$\overrightarrow{OM} = \frac{2\overrightarrow{OA} + \overrightarrow{OB}}{2+1} = \frac{1}{3}\left[2\begin{pmatrix}1\\-1\\2\end{pmatrix} + \begin{pmatrix}2\\4\\1\end{pmatrix}\right]$$

$$= \frac{1}{3}\begin{pmatrix}4\\2\\5\end{pmatrix}$$

(iii) Area of OAC

$$= \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OC}\right| = \frac{1}{2}\left|\begin{pmatrix}1\\-1\\2\end{pmatrix} \times \begin{pmatrix}-4\\2\\2\end{pmatrix}\right|$$

$$= \frac{1}{2}\left|\begin{pmatrix}-6\\-10\\-2\end{pmatrix}\right| = \frac{1}{2}\sqrt{140} = \sqrt{35}$$

7. (i) Second root $= re^{-i\theta}$

Quadratic factor

$= \left(z - re^{i\theta}\right)\left(z - re^{-i\theta}\right)$

$= z^2 - \left(re^{-i\theta} + re^{i\theta}\right)z + \left(re^{i\theta}\right)\left(re^{-i\theta}\right)$

$= z^2 - 2rz\cos\theta + r^2$ (shown)

(ii) $z^6 = -64 = 64e^{i\pi}$

$z = 64^{\frac{1}{6}}e^{i\left(\frac{\pi+2k\pi}{6}\right)}, k = 0, 1, 2, 3, 4, 5$

$z = 2e^{i\left(\frac{\pi+2k\pi}{6}\right)}, k = 0, 1, 2, 3, 4, 5$

When $k = 0$,
$z = 2e^{i\frac{\pi}{6}}$

When $k = 1$,
$z = 2e^{i\frac{\pi}{2}}$

When $k = 2$,
$z = 2e^{i\frac{5\pi}{6}}$

When $k = 3$,
$z = 2e^{-i\frac{5\pi}{6}}$

When $k = 4$,
$z = 2e^{-i\frac{\pi}{2}}$

When $k = 5$,
$z = 2e^{-i\frac{\pi}{6}}$

(iii) $z^6 + 64$

$= \left(z - 2e^{i\frac{\pi}{6}}\right)\left(z - 2e^{-i\frac{\pi}{6}}\right)\left(z - 2e^{i\frac{\pi}{2}}\right)\left(z - 2e^{-i\frac{\pi}{2}}\right)\left(z - 2e^{i\frac{5\pi}{6}}\right)\left(z - 2e^{-i\frac{5\pi}{6}}\right)$

$= \left(z^2 - 2(2)z\cos\frac{\pi}{6} + 2^2\right)\left(z^2 - 2(2)z\cos\frac{\pi}{2} + 2^2\right)\left(z^2 - 2(2)z\cos\frac{5\pi}{6} + 2^2\right)$

$= \left(z^2 - 2\sqrt{3}z + 4\right)\left(z^2 + 4\right)\left(z^2 + 2\sqrt{3}z + 4\right)$

8. (i) $\overrightarrow{AB} = \begin{pmatrix}-2\\3\\1\end{pmatrix} - \begin{pmatrix}1\\2\\4\end{pmatrix} = \begin{pmatrix}-3\\1\\-3\end{pmatrix}$

$\therefore l: \boldsymbol{r} = \begin{pmatrix}1\\2\\4\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\3\end{pmatrix}$

$p = \boldsymbol{r} \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix} = 17$

Let the point of intersection be R.

$\therefore OR = \begin{pmatrix}1+3\lambda\\2-\lambda\\4+3\lambda\end{pmatrix}$

Since R lies on P,

$\begin{pmatrix}1+3\lambda\\2-\lambda\\4+3\lambda\end{pmatrix} \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix} = 17$

$3 + 9\lambda - 2 + \lambda + 8 + 6\lambda = 17$

$\lambda = \frac{1}{2}$

$\therefore \overrightarrow{OR} = \begin{pmatrix}1+{}^{3}/_{2}\\2-{}^{1}/_{2}\\4+{}^{3}/_{2}\end{pmatrix} = \begin{pmatrix}{}^{5}/_{2}\\{}^{3}/_{2}\\{}^{11}/_{2}\end{pmatrix}$

Intersection between l and p is at $\left(\frac{5}{2}, \frac{3}{2}, \frac{11}{2}\right)$.

(ii) Let the acute angle required be θ.

$$\cos(90° - \theta) = \left|\frac{\begin{pmatrix}3\\-1\\2\end{pmatrix} \cdot \begin{pmatrix}3\\-1\\3\end{pmatrix}}{\left|\begin{pmatrix}3\\-1\\2\end{pmatrix}\right|\left|\begin{pmatrix}3\\-1\\3\end{pmatrix}\right|}\right|$$

$$= \frac{16}{\sqrt{19}\sqrt{14}}$$

$$\theta = 90° - \cos^{-1}\frac{16}{\sqrt{19}\sqrt{14}} = 78.8°$$

(iii) Distance

$$= \left|\frac{\begin{pmatrix}1\\2\\4\end{pmatrix} \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix}}{\left|\begin{pmatrix}3\\-1\\2\end{pmatrix}\right|} - \frac{17}{\left|\begin{pmatrix}3\\-1\\2\end{pmatrix}\right|}\right| = \left|\frac{9-17}{\sqrt{14}}\right|$$

$$= \frac{4}{7}\sqrt{14}$$

9. (i) From GC, $\alpha = 0.619$, $\beta = 1.512$

(ii) Assuming the sequence converges, when $n \to \infty$, let $x_n \to L$, $x_{n+1} \to L$

$\therefore L = \frac{1}{3}e^L$

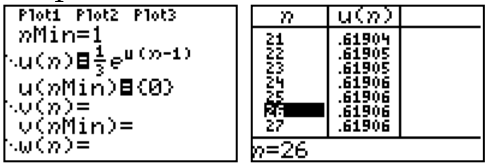
$3L = e^L \Rightarrow e^L - 3L = 0$

$\therefore$ from part (i), $L = \alpha$ or $L = \beta$

$\Rightarrow$ sequence converges to α or β.

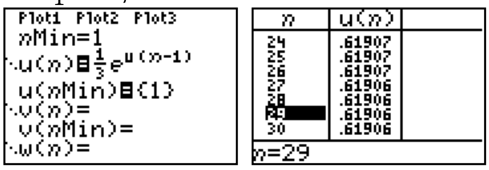
(iii) From GC,

When $x_1 = 0$,



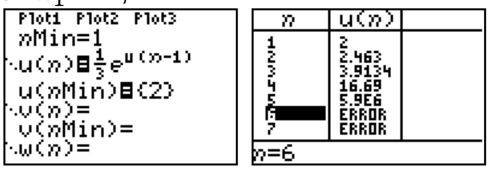
$\therefore$ sequence converges to 0.61906 (α)

When $x_1 = 1$,



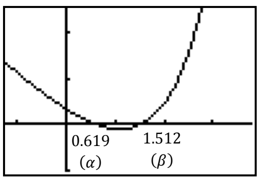
$\therefore$ sequence converges to 0.61906 (α)

When $x_1 = 2$,



$\therefore$ sequence diverges

(iv) Let $y = x_{n+1} - x_n = \frac{1}{3}e^{x_n} - x_n$



From graph, when $\alpha < x_n < \beta$,

$\frac{1}{3}e^{x_n} - x_n < 0$

$x_{n+1} - x_n < 0$

$x_{n+1} < x_n$ (shown)

when $x_n < \alpha$ or $x_n > \beta$,

$\frac{1}{3}e^{x_n} - x_n > 0$

$x_{n+1} - x_n > 0$

$x_{n+1} > x_n$ (shown)

(v) When $x_1 = 0$, $x_1 < \alpha$. So from part (iv), subsequent x's will increase. (A)

When $x_1 = 1$, $\alpha < x_1 < \beta$, so from part (iv), subsequent x's will decrease. (B)

(A) & (B) $\Rightarrow x_{n+1}$ converges to $0.619(\alpha)$

When $x_1 = 2$, $x_1 > \beta$. So from part (iv), subsequent x's will increase indefinitely.

$\therefore x_{n+1}$ diverges.

10. From the given information,

	AP	GC	
(1)	a	a	(1)
(4)	$a+3d$	ar	(2)
(6)	$a+5d$	ar^2	(3)

(a) $ar = a + 3d$

$d = \frac{ar-a}{3} = \frac{a}{3}(r-1)$

$ar^2 = a + 5d$

$ar^2 = a + \frac{5a}{3}(r-1)$

$r^2 = 1 + \frac{5}{3}(r-1) \quad (\because a \neq 0)$

$3r^2 = 3 + 5r - 5$

$3r^2 - 5r + 2 = 0$ (shown)

(b) $3r^2 - 5r + 2 = 0$

$(3r-2)(r-1) = 0$

$r = \frac{2}{3}$ or $r = 1$ (NA)

Since $|r| < 1$, GP is convergent

$S_\infty = \frac{a}{1-\frac{2}{3}} = 3a$

(c) $d = \frac{a}{3}\left(\frac{2}{3} - 1\right) = -\frac{a}{9}$

$S > 4a$

$\frac{n}{2}\left[2a + (n-1)\left(-\frac{a}{9}\right)\right] > 4a$

$\frac{n}{2}\left(2 + \frac{1-n}{9}\right) > 4 \quad (\because a > 0)$

$\frac{n}{2}\left(\frac{19-n}{9}\right) > 4$

$19n - n^2 > 72$

$n^2 - 19n + 72 < 0$

$5.228 < n < 13.772$

$6 \leq n \leq 13$

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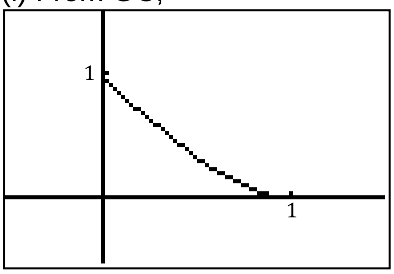
11. (i) From GC,



(ii)

$\frac{dx}{dt} = 2\cos t\,(-\sin t) = -2\sin t\cos t$

$\frac{dy}{dt} = 3\sin^2 t\cos t$

$\frac{dy}{dx} = \frac{3\sin^2 t\cos t}{-2\sin t\cos t} = -\frac{3}{2}\sin t$

At $(\cos^2\theta, \sin^3\theta)$,

$t = \theta$

$\frac{dy}{dx} = -\frac{3}{2}\sin\theta$

∴ Equation of tangent:

$y - \sin^3\theta = -\frac{3}{2}\sin\theta\,(x - \cos^2\theta)$

When $x = 0$,

$y - \sin^3\theta = -\frac{3}{2}\sin\theta\,(-\cos^2\theta)$

$y = \frac{3}{2}\sin\theta\cos^2\theta + \sin^3\theta$

$= \sin\theta\left(\frac{3}{2}\cos^2\theta + \sin^2\theta\right)$

When $y = 0$,

$-\sin^3\theta = -\frac{3}{2}\sin\theta\,(x - \cos^2\theta)$

$x - \cos^2\theta = \frac{2}{3}\sin^2\theta$

$x = \frac{2}{3}\sin^2\theta + \cos^2\theta$

∴ Area of ΔOQR

$= \frac{1}{2}\left[\sin\theta\left(\frac{3}{2}\cos^2\theta + \sin^2\theta\right)\right]\left[\frac{2}{3}\sin^2\theta + \cos^2\theta\right]$

$= \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\sin\theta\,(3\cos^2\theta + 2\sin^2\theta)(2\sin^2\theta + 3\cos^2\theta)$

$= \frac{1}{12}\sin\theta\,(3\cos^2\theta + 2\sin^2\theta)^2$

(shown)

(iii) At Q,

$x = 1 \Rightarrow t = 0$

At R,

$x = 0 \Rightarrow t = \frac{\pi}{2}$

Area

$= \int_0^1 y\,dx = \int_{\frac{\pi}{2}}^{0} \sin^3 t\,(-2\sin t\cos t)\,dt$

$= 2\int_0^{\frac{\pi}{2}} \cos t\sin^4 t\,dt$

(shown)

$u = \sin t \Rightarrow t = \sin^{-1} u$ $\frac{dt}{du} = \frac{1}{\sqrt{1-u^2}}$ When $t = 0$, $u = 0$ When $t = \frac{\pi}{2}$, $u = 1$ $\therefore$ Area $= 2\int_0^1 \sqrt{1-u^2}(u^4)\frac{1}{\sqrt{1-u^2}}\,du$ $= 2\int_0^1 u^4\,du$ $= 2\left[\frac{1}{5}u^5\right]_0^1 = \frac{2}{5}$	