INTEGRATION – TECHNIQUES

1. Default Strategy	• Manipulations, such as
	  ○ Algebraic (e.g. partial fractions, long division, completing square etc.)
	  ○ Trigonometric Identities
	• Standard Integrals
	• $f(x)$ $f'(x)$

2. Backup Strategy	By Parts
Only if instructed by question	By Substitution

1. (a) Find $\int \sin(2\theta)\cos(3\theta)\,d\theta$. [2]

   (b) Use the substitution $\theta = \sqrt{x}$ to find the exact value of $\int_{\frac{\pi}{2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\,d\theta$. [5]

[JJC]

[Ans: (a) $\frac{1}{2}\cos(\theta) - \frac{1}{10}\cos(5\theta) + C$ (b) $-\frac{1}{2} - \frac{\pi}{4}$]

2. (a) Find $\int \frac{2-x}{4+x^2}\,dx$. [3]

(b) Use the substitution $x = \tan y$ to find the exact value of $\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$. [4]

(c) Write down $\int x^2 e^{x^3}\,dx$. Hence find $\int x^5 e^{x^3}\,dx$. [4]

[CJC]

[Ans: (a) $\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\ln(4+x^2) + C$ (b) $\ln\left(\sqrt{2}+1\right)$ (c) $\frac{1}{3}e^{x^3} + C$; $\frac{1}{3}x^3 e^{x^3} \frac{1}{3}e^{x^3} + C$]

3. (a) (i) Find $\int \cos 3x \cos x dx$. [3]

(ii) Hence, find the exact value of $\int_0^{\frac{\pi}{4}} x \cos 3x \cos x dx$. [4]

(b) Find, in terms of a, $\int_{-a}^{a} \left| x\left(x-2a\right) \right| dx$, where a is a positive constant. [3]

[YJC]

[Ans: (a)(i) $\frac{1}{8}\sin 4x + \frac{1}{4}\sin 2x + C$ (ii) $\frac{\pi-3}{16}$ (b) $2a^3$]

4. (i) Find $\int \frac{x}{\left(1+x^2\right)^2} dx$. [2]

(ii) By using the substitution $x = \tan\theta$, show that

$$\int \frac{1}{\left(1+x^2\right)^2} dx = k\left(\frac{x}{1+x^2} + \tan^{-1} x\right) + c,$$

where c is an arbitrary constant, and k is a constant to be determined. [5]

(iii) Hence find $\int \frac{x^2}{\left(1+x^2\right)^2} dx$. [3]

(iv) Using all of the above, find $\int \frac{x^2+2x+5}{\left(1+x^2\right)^2} dx$, simplifying your answer. [2]

[ACJC]

[Ans: (i) $-\frac{1}{2(1+x^2)} + c$ (ii) $\frac{1}{2}\left(\frac{x}{1+x^2} + \tan^{-1} x\right) + c$, $k = \frac{1}{2}$ (iii) $\frac{1}{2}\left(\tan^{-1} x - \frac{x}{1+x^2}\right) + c$ (iv) $3\tan^{-1} x + \frac{2x-1}{1+x^2} + c$]

5. (i) Find $\frac{d}{dx}\tan^2 x$. Hence evaluate $\int_0^{\frac{1}{4}\pi} \sec^2 x \tan x e^{\tan^2 x} dx$, leaving your answer in exact form. [3]

   (ii) By expressing $1+72x-32x^3$ as $1+mx\left(9-4x^2\right)$ where m is a constant, find

   $$\int \frac{1+72x-32x^3}{\sqrt{9-4x^2}} dx .$$ [2]

[DHS]

[Ans: (i) $\frac{1}{2}(e-1)$ (ii) $\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)-\frac{2}{3}(9-4x^2)^{\frac{3}{2}}+C$]

6. By writing $\sec^3 x = \sec x \sec^2 x$, find $\int \sec^3 x dx$.

   Hence find the exact value of $\int_0^{\tan^{-1}2} \sec^3 x dx$. [6]

[HCI]

[Ans: $\frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$; $\sqrt{5} + \frac{1}{2}\ln(\sqrt{5}+2)$]

7. (i) Find $\int n\cos^{-1}(nx)\,dx$, where n is a positive constant. [3]

   (ii) Hence find the exact value of $\int_0^{\frac{1}{2n}} n\cos^{-1}(nx)\,dx$. [2]

[IJC]

[Ans: (i) $(nx)\cos^{-1}(nx) - \sqrt{(1-n^2x^2)} + C$ (ii) $\frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1$]

8. (a) (i) Evaluate $\int \frac{2x-4}{x^2-2x+4}\,dx$. [3]

(ii) Without the use of a graphic calculator, evaluate $\int_1^4 \frac{|2x-4|}{x^2-2x+4}\,dx$, leaving your answer in logarithmic form. [4]

(b) Given that $\frac{d}{dx}\left(\frac{1}{\cos^2 x}\right)=\frac{2\sin x}{\cos^3 x}$, evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^3 x}\,dx$, leaving your answer in exact form. [3]

[NYJC]

[Ans: (a)(i) $\ln(x^2-2x+4)-\frac{2}{\sqrt{3}}\tan^{-1}\frac{x-1}{\sqrt{3}}+C$ (ii) $2\ln\left(\frac{3}{2}\right)$ (b) $\frac{1}{\sqrt{2}}-\frac{1}{2}\ln\left(\sqrt{2}+1\right)$]

9. (i) Find $\int e^x \cos nx dx$, where n is a positive integer. [4]

   (ii) Hence, without the use of a calculator, find $\int_{\pi}^{2\pi} e^x \cos nx dx$ when n is odd. [3]

[SAJC]

[Ans: (i) $\frac{ne^x}{1+n^2}\left(\sin nx + \frac{\cos nx}{n}\right) + c$ (ii) $\left(\frac{1}{1+n^2}\right)(e^{2\pi} + e^{\pi})$]

10. (a) Using the substitution $u = 2x+3$, find $\int \frac{x}{(2x+3)^3}\,dx$ in the form $-\frac{Px+Q}{R(2x+3)^2}+c$, where P, Q and R are positive integers to be determined. [3]

Hence find $\int \frac{x\ln(4x+3)}{(2x+3)^3}\,dx$. [3]

(b) Find $\int \sin 4x \cos 6x\,dx$. [2]

Hence or otherwise, find $\int e^x \sin 4e^x \cos 6e^x\,dx$. [1]

[SRJC]

[Ans: (a) $-\frac{4x+3}{8(2x+3)^2}+c$; $-\frac{(4x+3)\ln(4x+3)+2(2x+3)}{8(2x+3)^2}+c$ (b) $-\frac{1}{20}\cos 10x+\frac{1}{4}\cos 2x+c$; $-\frac{1}{20}\cos 10e^x+\frac{1}{4}\cos 2e^x+c$]